An axiomatization of bilateral state-based modal logic

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Bilateral state-based modal logic (*BSML*)—introduced to model *neglect-zero effects* and to account for free choice inferences and related phenomena.

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Neglect-zero tendency: tendency to disregard structures that verify sentences by virtue of some empty configuration.

[■, ■, ■]	[■, □, ■]	[ ]; [△, △, △]; [◇, ▲, ♠]
(a) Verifier	(b) Falsifier	(c) Zero-models

Models for the sentence *Every square is black.*

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Neglect-zero tendency: tendency to disregard structures that verify sentences by virtue of some empty configuration.

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[■, □, ■] (b) Falsifier

[ ]; [△, △, △]; [◇, ▲, ♠] (c) Zero-models

Models for the sentence *Every square is black.*

We present a natural deduction system for *BSML*.

We also examine expressive power: We have no expressive completeness result for *BSML*; we introduce and axiomatize two expressively complete extensions.

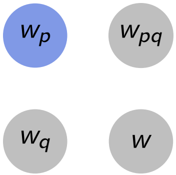
Bilateral State-based Modal Logic

$$M = (W, R, V)$$

standard Kripke semantics

$$M, w \vDash \phi$$

$$w \in W$$

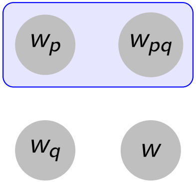


$$w_p \vDash p$$

state-based/team semantics

$$M, s \vDash \phi$$

$$s \subseteq W$$



$$\{w_p, w_{pq}\} \vDash p$$

Bilateralism

"ϕ is assertable in s" $\quad s \models \phi$

"ϕ is rejectable in s" $\quad s \dashv \phi$

Bilateral negation

$$s \models \neg\phi \iff s \dashv \phi$$

Syntax of *BSML*

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

Semantics ($\vDash$)

$$
\begin{aligned}
s \vDash p &\iff \forall w \in s : w \in V(p)\\
s \vDash \neg\phi &\iff s \dashv \phi\\
s \vDash \phi \wedge \psi &\iff s \vDash \phi \text{ and } s \vDash \psi\\
s \vDash \phi \vee \psi &\iff \exists t, t' : t \cup t' = s \text{ and } t \vDash \phi \text{ and } t' \vDash \psi\\
s \vDash \Diamond\phi &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi\\
s \vDash \mathrm{NE} &\iff s \neq \varnothing
\end{aligned}
$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

Syntax of *BSML*

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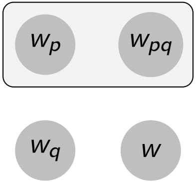
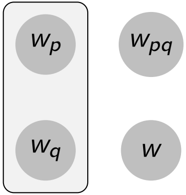
\end{array}$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

$$s \models p \iff \forall w \in s : w \in V(p)$$



(a) $s \models p$



(b) $s \not\models p$

Syntax of *BSML*

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

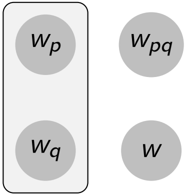
Semantics ($\models$)

$$
\begin{array}{lcl}
s \models p & \iff & \forall w \in s : w \in V(p) \\
s \models \neg\phi & \iff & s \,=\!\!\mid\, \phi \\
s \models \phi \wedge \psi & \iff & s \models \phi \text{ and } s \models \psi \\
s \models \phi \vee \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi \\
s \models \Diamond\phi & \iff & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi \\
s \models \mathrm{NE} & \iff & s \neq \varnothing
\end{array}
$$

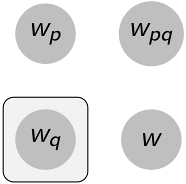
$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

Tensor disjunction ∨

$$s \vDash \phi \vee \psi \iff \exists t, t' : \begin{array}{ll} t \cup t' = s & \text{and} \\ t \vDash \phi & \text{and} \\ t' \vDash \psi & \end{array}$$



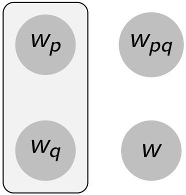
(a) $s \vDash p \vee q$



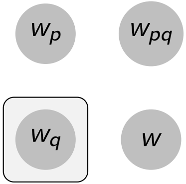
(b) $s \vDash p \vee q$

The non-emptiness atom NE

$$s \models \mathrm{NE} \iff s \neq \varnothing$$



(a) $s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

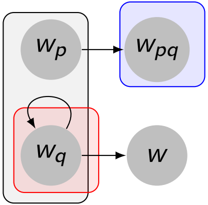


(b) $s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

The modality $\Diamond$

$$R[w] = \{v \in W \mid wRv\}$$

$$s \vDash \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \text{ and } t \vDash \phi$$



$s \vDash \Diamond q$ since

$\{w_q\} \subseteq R[w_q]$
$\{w_q\} \vDash q$

and

$\{w_{pq}\} \subseteq R[w_p]$
$\{w_{pq}\} \vDash q$

We can model the neglect-zero tendency in *BSML* using the *pragmatic enrichment function* $[]^+$

$$
\begin{aligned}
p^+ &:= p \wedge \mathrm{NE} \\
(\neg\phi)^+ &:= \neg\phi^+ \wedge \mathrm{NE} \\
(\phi \wedge \psi)^+ &:= (\phi^+ \wedge \psi^+) \wedge \mathrm{NE} \\
(\phi \vee \psi)^+ &:= (\phi^+ \vee \psi^+) \wedge \mathrm{NE} \\
(\Diamond\phi)^+ &:= \Diamond\phi^+ \wedge \mathrm{NE}
\end{aligned}
$$

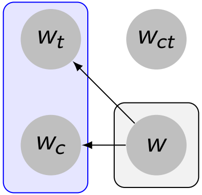
Free choice (FC) inferences:

You may have coffee or tea.
⇝You may have coffee and you may have tea.

$$(\Diamond(c \vee t))^+ \vDash \Diamond c \wedge \Diamond t$$

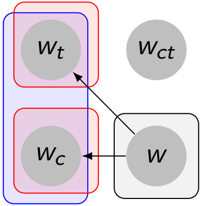
i.e. $\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \vDash \Diamond c \wedge \Diamond t$

$$\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \vDash \Diamond c \wedge \Diamond t$$



$\{w\} \vDash \Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE}))$ since

$$\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \models \Diamond c \wedge \Diamond t$$



$\{w\} \models \Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE}))$ since $\{w_c\} \models c$ and $\{w_t\} \models t$

for the same reason, $\{w\} \models \Diamond c \wedge \Diamond t$

Extensions:

$BSML^{\mathbb{W}}$: *BSML* with the global/inquisitive disjunction $\mathbb{W}$

$$s \models \phi \mathbb{W} \psi \quad \Longleftrightarrow \quad s \models \phi \text{ or } s \models \psi$$

$BSML^{\oslash}$: *BSML* with the emptiness operator $\oslash$

$$s \models \oslash\phi \quad \Longleftrightarrow \quad s \models \phi \text{ or } s = \varnothing$$

Semantics ($\dashv$)

$$
\begin{array}{lcl}
s \dashv p & \iff & \forall w \in s : w \notin V(p) \\
s \dashv \neg\phi & \iff & s \models \phi \\
s \dashv \phi \wedge \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \dashv \phi \text{ and } t' \dashv \psi \\
s \dashv \phi \vee \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
s \dashv \phi \mathbin{\mathbb{V}} \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
s \dashv \Diamond\phi & \iff & \forall w \in s : R[w] \dashv \phi \\
s \dashv \mathrm{NE} & \iff & s = \varnothing \\
s \dashv \oslash\phi & \iff & s \dashv \phi
\end{array}
$$

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s \dashv \phi \vee \psi &\iff s \dashv \phi \text{ and } s \dashv \psi\\
s \dashv \phi \mathbin{\mathbb{W}} \psi &\iff s \dashv \phi \text{ and } s \dashv \psi\\
s \dashv \Diamond\phi &\iff \forall w \in s : R[w] \dashv \phi\\
s \dashv \mathrm{NE} &\iff s = \varnothing\\
s \dashv \oslash\phi &\iff s \dashv \phi
\end{aligned}
$$

$\neg\alpha$ behaves classically when α is classical (no NE, $\mathbb{W}$, $\oslash$)

$\Box := \neg\Diamond\neg$

$s \vDash \Box\phi \iff \forall w \in s : R[w] \vDash \phi$

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi & \neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi\\
\neg\mathrm{NE} &\equiv p \wedge \neg p & \neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi\\
\neg\oslash\phi &\equiv \neg\phi & \neg(\phi \mathbin{\mathbb{W}} \psi) &\equiv \neg\phi \wedge \neg\psi\\
\neg\Diamond\phi &\equiv \Box\neg\phi & &
\end{aligned}
$$

Weak contradiction $\bot := p \wedge \neg p$. $s \vDash \bot \iff s = \varnothing$.

Strong contradiction $\bot\!\!\!\bot := \bot \wedge \mathrm{NE}$. $s \vDash \bot\!\!\!\bot$ is never true.

Weak contradiction $\bot := p \wedge \neg p$. $s \models \bot \iff s = \varnothing$.

Strong contradiction $\bot\!\!\!\bot := \bot \wedge \mathrm{NE}$. $s \models \bot\!\!\!\bot$ is never true.

$$\models \bot \mathbin{\mathbb{W}} \mathrm{NE} \qquad \oslash\, \phi \equiv \bot \mathbin{\mathbb{W}} \phi \qquad \models \oslash \mathrm{NE}$$

Closure properties

ϕ is *downward closed*: $[M, s \models \phi \text{ and } t \subseteq s] \implies M, t \models \phi$

ϕ is *union closed*: $[M, s \models \phi \text{ for all } s \in S \neq \varnothing] \implies M, \bigcup S \models \phi$

ϕ has the *empty state property*: $M, \varnothing \models \phi$ for all M

ϕ is *flat*: $M, s \models \phi \iff M, \{w\} \models \phi$ for all $w \in s$

flat $\iff$ downward closed & union closed & empty state property

Closure properties

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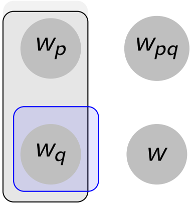
ϕ is *flat*: $M, s \vDash \phi \iff M, \{w\} \vDash \phi$ for all $w \in s$

flat $\iff$ downward closed & union closed & empty state property

Formulas in classical modal logic *ML* (no NE, $\mathbin{\mathbb{W}}$, $\oslash$) are flat and their state semantics coincide with their standard semantics on singletons $\{w\}$:

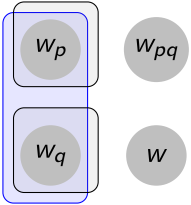
$$s \vDash \alpha \iff \forall w \in s : \{w\} \vDash \alpha \iff \forall w \in s : w \vDash \alpha$$

Formulas with NE may lack downward closure and the empty state property:



$$\{w_p, w_q\} \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$
$$\{w_q\} \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$

Formulas with ⩔ may lack union closure:



$$\{w_p\} \models p \mathbin{⩔} q$$
$$\{w_q\} \models p \mathbin{⩔} q$$
$$\{w_p, w_q\} \not\models p \mathbin{⩔} q$$

Expressive Power

We show $BSML^{\mathbb{W}}$ and $BSML^{\oslash}$ are expressively complete and:

$$ML < BSML < BSML^{\oslash} < BSML^{\mathbb{W}}$$

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$$ML < BSML < BSML^{\oslash} < BSML^{\mathbb{W}}$$

Fix a finite set of proposition symbols Φ

Pointed state model: (M, s) where M is a model over Φ; s is a state on M

state property: set of pointed state models

$||\phi|| := \{(M, s) \mid M, s \vDash \phi\}$

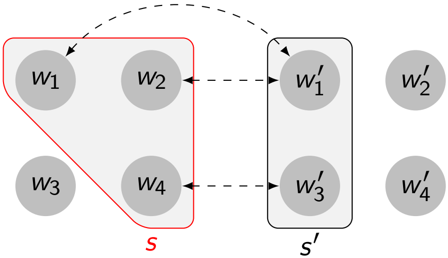
Theorem

$$\{||\phi|| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under state } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

state bisimulation:

$s \leftrightarrows_k s' :\Longleftrightarrow$

forth: $\forall w \in s : \exists w' \in s' : w \leftrightarrows_k w'$

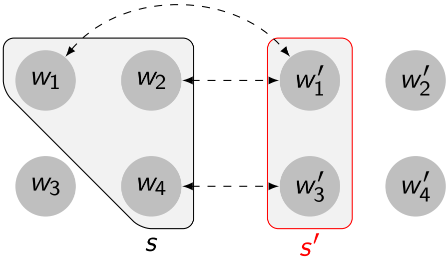


state bisimulation:

$s \leftrightarroweq_k s' :\Longleftrightarrow$

forth: $\forall w \in s : \exists w' \in s' : w \leftrightarroweq_k w'$

back: $\forall w' \in s' : \exists w \in s : w \leftrightarroweq_k w'$



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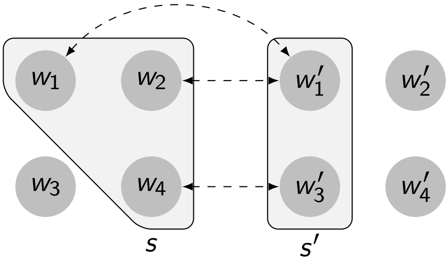

modal depth $md(\phi)$: measure of maximum nesting of modalities in ϕ. E.g. $md(\Diamond p \vee \Box(q \wedge \Diamond p)) = 2$

$s \equiv^k s'$: $\Longleftrightarrow$ $s \vDash \phi$ iff $s' \vDash \phi$ for all ϕ with $md(\phi) \leq k$

Theorem (bisimulation invariance)

$$s \leftrightarroweq_k s' \Longrightarrow s \equiv^k s'$$

Property P is *invariant under state k-bisimulation*:

$$[(M,s) \in P \text{ and } M,s \leftrightarroweq_k M',s'] \implies (M',s') \in P$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under state } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Characteristic formulas for worlds (Hintikka formulas)

$$\chi^0_{M,w} := \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

$$\chi^{k+1}_{M,w} := \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v}$$

$$w' \vDash \chi^k_w \iff w \leftrightarroweq_k w'$$

Characteristic formulas for worlds (Hintikka formulas)

$$\chi^0_{M,w} := \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

$$\chi^{k+1}_{M,w} := \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v}$$

$$w' \vDash \chi^k_w \iff w \leftrightarroweq_k w'$$

Characteristic formulas for states:

$$\theta^k_{M,s} := \bot \quad \text{if } s = \varnothing \quad (\bot := p \wedge \neg p)$$
$$\theta^k_{M,s} := \bigvee_{w \in s} (\chi^k_{M,w} \wedge \mathrm{NE}) \quad \text{if } s \neq \varnothing$$

$$s' \vDash \theta^k_s \iff s \leftrightarroweq_k s'$$

Characteristic formulas for properties

for P invariant under k-bisimulation:

$$M', s' \models \bigvee_{(M,s)\in P} \theta_s^k \iff (M', s') \in P$$

Theorem

$$\{||\phi|| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under state } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

This also yields a disjunctive normal form for formulas of $BSML^{\mathbb{W}}$:

$$\phi \equiv \bigvee_{(M,s)\in||\phi||} \theta_s^{md(\phi)}$$

Property P is *union closed*:

$$\{(M, s_i) \mid i \in I \neq \varnothing\} \subseteq P \implies (M, \bigcup_{i \in I} s_i) \subseteq P$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$

$$=$$

$$\mathbb{U} := \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

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Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$
$$=$$
$$\mathbb{U} := \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

$BSML$ is union closed, but not expressively complete for $\mathbb{U}$

Example: $\|(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})\| \cup \|\bot\| \in \mathbb{U}$ but not expressible in $BSML$

in $BSML^{\oslash}$: $\oslash((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$

$$s' \models \theta_s^k \iff s \leftrightarroows_k s'$$

$$s' \models \oslash\theta_s^k \iff s \leftrightarroows_k s' \text{ or } s = \varnothing$$

Characteristic formulas for union-closed properties with the empty state property:

$$M', s' \models \bigvee_{(M,s)\in P} \oslash\theta_s^k \iff (M', s') \in P$$

Characteristic formulas for union-closed properties without the empty state property:

$$M', s' \models \mathrm{NE} \wedge \bigvee_{(M,s)\in P} \oslash\theta_s^k \iff (M', s') \in P$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$
$$=$$
$$\{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{P \mid P \text{ is invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

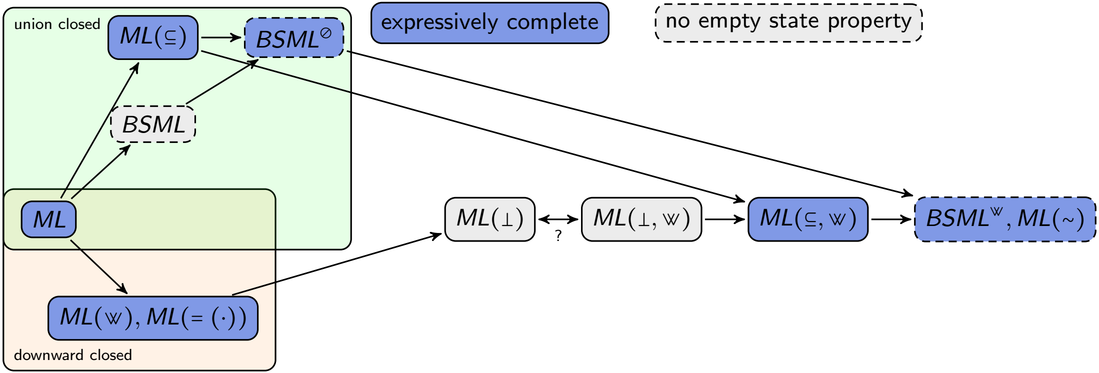
Normal form: $\phi \equiv \bigvee\!\!\!\!\bigvee_{(M,s)\in\|\phi\|} \theta_s^{md(\phi)}$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$
$$=$$
$$\{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Normal forms $\phi \equiv \bigvee_{(M,s)\in\|\phi\|} \oslash\theta_s^{md(\phi)}$ or $\phi \equiv \mathrm{NE} \wedge \bigvee_{(M,s)\in\|\phi\|} \oslash\theta_s^{md(\phi)}$

Expressive powers compared:



$=(\cdot)$: extended dependence atoms: $s \models =\!(\alpha_1, \ldots, \alpha_n, \beta) :\Longleftrightarrow$
$\forall w, w' \in s : (w \models \alpha_i \iff w' \models \alpha_i$ for all $i \in \{1, \ldots, n\})$ implies $w \models \beta \iff w' \models \beta$

$\subseteq$: extended inclusion atoms: $s \models \alpha_1, \ldots \alpha_n \subseteq \beta_1, \ldots, \beta_n :\Longleftrightarrow$
$\forall w \in s : \exists v \in s : w \models \alpha_i \iff v \models \beta_i$ for all $i \in \{1, \ldots, n\}$

$\perp$: extended independence atoms: $s \models \alpha_1, \ldots \alpha_n \perp_{\gamma_1, \ldots, \gamma_m} \beta_1, \ldots, \beta_l :\Longleftrightarrow$
$\forall w, w' \in s : (w \models \gamma_i \iff w' \models \gamma_i)$ implies $\exists v \in s : (w \models \alpha_i \iff v \models \alpha_i)$ and $(w' \models \beta_i \iff v \models \beta_i)$ and $(w \models \gamma_i \iff v \models \gamma_i)$

$\sim$: Boolean negation: $s \models \sim \phi :\Longleftrightarrow s \not\models \phi$

System for $BSML^{\mathbb{W}}$

α and β: classical formulas (no NE or $\mathbb{W}$ or $\oslash$).

¬-introduction	¬-elimination
$$\dfrac{\begin{array}{c}[\alpha]\\ D^*\\ \bot\end{array}}{\neg\alpha}\,\neg I(*)$$	$$\dfrac{\begin{array}{cc}D_1 & D_2\\ \alpha & \neg\alpha\end{array}}{\beta}\,\neg E$$

$(*)$ The undischarged assumptions in D^* do not contain NE.

∧-introduction

$$\dfrac{\begin{matrix} D_1 \\ \phi \end{matrix} \qquad \begin{matrix} D_2 \\ \psi \end{matrix}}{\phi \wedge \psi} \wedge I$$

∧-elimination

$$\dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\phi} \wedge E \qquad \dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\psi} \wedge E$$

⩔-introduction

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \Vee \psi} \Vee I \qquad \dfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi \Vee \psi} \Vee I$$

⩔-elimination

$$\dfrac{\begin{matrix} D \\ \phi \Vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix}}{\chi} \Vee E$$

∨-weak introduction

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \vee \psi} \vee I(**)$$

∨-weakening

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \vee \phi} \vee W$$

∨-weak elimination

$$\dfrac{\begin{matrix} D \\ \phi \vee \psi \end{matrix} \quad \begin{matrix} [\phi] \\ D_1^* \\ \chi \end{matrix} \quad \begin{matrix} [\psi] \\ D_2^* \\ \chi \end{matrix}}{\chi} \vee E(*, \dagger)$$

∨-weak substitution

$$\dfrac{\begin{matrix} D \\ \phi \vee \psi \end{matrix} \quad \begin{matrix} [\psi] \\ D_1^* \\ \chi \end{matrix}}{\phi \vee \chi} \vee Sub(*)$$

$(*)$ The undischarged assumptions in D_1^*, D_2^* do not contain NE.
$(**)$ ψ does not contain NE.
$(\dagger)$ χ does not contain $\mathbb{W}$, or χ is of the form $\Diamond\eta$ or $\Box\eta$.

∨-commutativity

$$\dfrac{\begin{matrix} D \\ \phi \vee \psi \end{matrix}}{\psi \vee \phi}\, \mathit{Com}\vee$$

∨⩔-distributivity

$$\dfrac{\begin{matrix} D \\ \phi \vee (\psi \mathbin{⩔} \chi) \end{matrix}}{(\phi \vee \psi) \mathbin{⩔} (\phi \vee \chi)}\, \mathit{Distr} \vee ⩔$$

⊥-elimination	⫫-contraction	NE-introduction
$$\dfrac{\begin{matrix} D \\ \phi \vee \bot \end{matrix}}{\phi}\ \bot E$$	$$\dfrac{\begin{matrix} D \\ \phi \vee \perp\!\!\!\perp \end{matrix}}{\psi}\ \perp\!\!\!\perp Ctr$$	$$\dfrac{}{\bot \mathbin{\mathbb{W}} \mathrm{NE}}\ \mathrm{NE}I$$

$\neg\mathrm{NE}$ elimination

$$\frac{\begin{matrix} D \\ \neg\mathrm{NE} \end{matrix}}{\bot}\ \neg\mathrm{NE}E$$

Double $\neg$ elimination

$$\frac{\begin{matrix} D \\ \neg\neg\phi \end{matrix}}{\phi}\ DN$$

De Morgan's laws

$$\frac{\begin{matrix} D \\ \neg(\phi \wedge \psi) \end{matrix}}{\neg\phi \vee \neg\psi}\ DM_{\wedge} \qquad \frac{\begin{matrix} D \\ \neg(\phi \vee \psi) \end{matrix}}{\neg\phi \wedge \neg\psi}\ DM_{\vee} \qquad \frac{\begin{matrix} D \\ \neg(\phi \mathbin{⩔} \psi) \end{matrix}}{\neg\phi \wedge \neg\psi}\ DM$$

Modal rules:

$\Diamond$-monotonicity

$$\dfrac{\begin{matrix}[\phi] & \\ D' & D \\ \psi & \Diamond\phi\end{matrix}}{\Diamond\psi}\ \Diamond Mon(*)$$

$\Box$-monotonicity

$$\dfrac{\begin{matrix}[\phi_1]\dots[\phi_n] & & & \\ D' & D_1 & & D_n \\ \psi & \Box\phi_1 & \dots & \Box\phi_n\end{matrix}}{\Box\psi}\ \Box Mon(*)$$

$\Diamond\Box$-interaction

$$\dfrac{\begin{matrix}D \\ \neg\Diamond\phi\end{matrix}}{\Box\neg\phi}\ Inter\Diamond\Box$$

($*$) D' does not contain undischarged assumptions.

$\Diamond \mathbin{\mathbb{W}} \vee$-conversion

$$\frac{\begin{array}{c} D \\ \Diamond(\phi \mathbin{\mathbb{W}} \psi) \end{array}}{\Diamond\phi \vee \Diamond\psi} \; Conv \Diamond \mathbin{\mathbb{W}} \vee$$

$\Box \mathbin{\mathbb{W}} \vee$-conversion

$$\frac{\begin{array}{c} D \\ \Box(\phi \mathbin{\mathbb{W}} \psi) \end{array}}{\Box\phi \vee \Box\psi} \; Conv \Box \mathbin{\mathbb{W}} \vee$$

$\Diamond$-separation

$$\dfrac{\begin{matrix} D \\ \Diamond(\phi \vee (\psi \wedge \mathrm{NE})) \end{matrix}}{\Diamond \psi} \; \Diamond \mathit{Sep}$$

$\Diamond$-join

$$\dfrac{\begin{matrix} D_1 & & D_2 \\ \Diamond \phi & & \Diamond \psi \end{matrix}}{\Diamond(\phi \vee \psi)} \; \Diamond \mathit{Join}$$

$\Box$-instantiation

$$\dfrac{\begin{matrix} D \\ \Box(\phi \wedge \mathrm{NE}) \end{matrix}}{\Diamond \phi} \; \Box \mathit{Inst}$$

$\Box\Diamond$-join

$$\dfrac{\begin{matrix} D_1 & & D_2 \\ \Box \phi & & \Diamond \psi \end{matrix}}{\Box(\phi \vee \psi)} \; \Box \Diamond \mathit{Join}$$

$s \models \Diamond \phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $t \models \phi$

$s \models \Box \phi \iff \forall w \in s : R[w] \models \phi$

Completeness

We use the disjunctive normal form:

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Lemma: $$\phi \in BSML^{\mathbb{W}} \implies \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s) \in P} \theta_s^k$$

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$$\phi \vDash \psi$$

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$$\phi \vDash \psi \implies \bigvee\!\!\!\bigvee_{(M,s)\in P} \theta^k_s \vDash \bigvee\!\!\!\bigvee_{(N,t)\in Q} \theta^k_t$$

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$$\phi \models \psi \implies \bigvee\!\!\!\!\bigvee_{(M,s)\in P} \theta_s^k \models \bigvee\!\!\!\!\bigvee_{(N,t)\in Q} \theta_t^k$$

$$\implies \forall (M,s) \in P : \exists (N,t) \in Q : \quad s \leftrightarroows_k t$$

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$$\implies \forall (M,s) \in P : \exists (N,t) \in Q : \quad \begin{array}{c} s \leftrightarrows_k t \\ \theta_s^k \dashv\vdash \theta_t^k \end{array}$$

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$$\implies \forall (M,s) \in P : \exists (N,t) \in Q : \quad \begin{array}{l} s \leftrightarrows_k t \\ \theta^k_s \dashv\vdash \theta^k_t \end{array}$$

$$\implies \bigvee\!\!\!\!\bigvee_{(M,s)\in P} \theta^k_s \vdash \bigvee\!\!\!\!\bigvee_{(N,t)\in Q} \theta^k_t$$

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Lemma: $\phi \in BSML^{\mathbb{W}} \implies \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s) \in P} \theta_s^k$

$$\phi \vDash \psi \implies \bigvee_{(M,s) \in P} \theta_s^k \vDash \bigvee_{(N,t) \in Q} \theta_t^k$$

$$\implies \forall (M,s) \in P : \exists (N,t) \in Q : \quad \begin{array}{l} s \leftrightarrows_k t \\ \theta_s^k \dashv\vdash \theta_t^k \end{array}$$

$$\implies \bigvee_{(M,s) \in P} \theta_s^k \vdash \bigvee_{(N,t) \in Q} \theta_t^k \implies \phi \vdash \psi$$

System for $BSML^{\oslash}$

Omit $\mathbb{W}$-rules and add:

$$\oslash\phi \equiv \phi \mathbin{\mathbb{W}} \bot \qquad \neg \oslash \phi \equiv \neg\phi$$

$BSML^{\oslash}$	$BSML^{\mathbb{W}}$
$\oslash$-introduction: $\dfrac{\overset{D}{\bot}}{\oslash\phi}\oslash I$ $\quad$ $\dfrac{\overset{D}{\phi}}{\oslash\phi}\oslash I$	$\mathbb{W}$-introduction: $\dfrac{\overset{D}{\phi}}{\phi \mathbin{\mathbb{W}} \psi}\mathbb{W} I$ $\quad$ $\dfrac{\overset{D}{\psi}}{\phi \mathbin{\mathbb{W}} \psi}\mathbb{W} I$
$\oslash\mathrm{NE}$-introduction: $\dfrac{}{\oslash\mathrm{NE}}\oslash\mathrm{NE} I$	NE-introduction: $\dfrac{}{\bot \mathbin{\mathbb{W}} \mathrm{NE}}\mathrm{NE} I$
$\neg\oslash$-elimination: $\dfrac{\overset{D}{\neg \oslash \phi}}{\neg\phi}\neg \oslash E$ (double line)	

$[\chi, m]_\phi$: the specific occurrence of the formula χ beginning at the m-th symbol of ϕ

$\phi(\psi/[\chi, m])$: the result of replacing $[\chi, m]$ in ϕ (if it exists) with ψ

$[\chi, m]_\phi$: the specific occurrence of the formula χ beginning at the m-th symbol of ϕ

$\phi(\psi/[\chi, m])$: the result of replacing $[\chi, m]$ in ϕ (if it exists) with ψ

$[\chi, m]$ is $\mathbb{V}$-distributive in ϕ: $[\chi, m]$ is not in the scope of any $\neg$, $\Diamond$ or $\Box$ in ϕ

$\mathbb{V}$ distributes over $\wedge$, $\vee$, $\mathbb{V}$, and $\oslash$, but not over $\neg$, $\Diamond$, or $\Box$.

So if $[\chi, m]$ is $\mathbb{V}$-distributive in ϕ and $\chi \equiv \mathbb{V}_{i \in I} \chi_i$ then $\phi \equiv \phi(\mathbb{V}_{i \in I} \chi_i/[\chi, m])$.

Given $[\oslash\psi, m]$ $\mathbin{\text{w}}$-distributive in ϕ, we want to be able to derive all entailments of ϕ that follow from $\oslash\psi \equiv \psi \mathbin{\text{w}} \bot$ and the fact that $\vee$, $\wedge$ and $\oslash$ distribute over $\mathbin{\text{w}}$—for instance, since $\oslash\psi \vee \chi \equiv (\bot \mathbin{\text{w}} \psi) \vee \chi \equiv (\bot \vee \chi) \mathbin{\text{w}} (\phi \vee \chi)$, if $\bot \vee \chi \vdash \eta$ and $\psi \vee \chi \vdash \eta$ we want $\oslash\psi \vee \chi \vdash \eta$.

$BSML^{\oslash}$	$BSML^{\mathbin{\text{w}}}$
$\oslash$-elimination	$\mathbin{\text{w}}$-elimination

$BSML^{\oslash}$: $\oslash$-elimination

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi \end{array} \qquad \begin{array}{c} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi \end{array}}{\chi} \oslash E(*)$$

$(*)$ $[\oslash\psi, m]$ is $\mathbin{\text{w}}$-distributive in ϕ.

$BSML^{\mathbin{\text{w}}}$: $\mathbin{\text{w}}$-elimination

$$\dfrac{\begin{array}{c} D \\ \phi \mathbin{\text{w}} \psi \end{array} \qquad \begin{array}{c} [\phi] \\ D_1 \\ \chi \end{array} \qquad \begin{array}{c} [\psi] \\ D_2 \\ \chi \end{array}}{\chi} \mathbin{\text{w}} E$$

$BSML^{\oslash}$

$\Diamond\oslash$-elimination

$$\dfrac{\begin{array}{c} D \\ \Diamond\phi \end{array} \qquad \begin{array}{c} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi_1 \end{array} \qquad \begin{array}{c} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi_2 \end{array}}{\Diamond\chi_1 \vee \Diamond\chi_2}\ \Diamond\oslash E(*)$$

$\Box\oslash$-elimination

$$\dfrac{\begin{array}{c} D \\ \Box\phi \end{array} \qquad \begin{array}{c} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi_1 \end{array} \qquad \begin{array}{c} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi_2 \end{array}}{\Box\chi_1 \vee \Box\chi_2}\ \Box\oslash E(*)$$

$(*)$ $[\oslash\psi, m]$ is ⩖-distributive in ϕ.
D_1, D_2 do not contain undischarged assumptions.

(Here $[\oslash\psi, m]$ must to be ⩖-distributive in ϕ, not in $\Diamond\phi / \Box\phi$.)

$BSML^{⩖}$

$\Diamond$ ⩖ $\vee$-conversion

$$\dfrac{\begin{array}{c} D \\ \Diamond(\phi ⩖ \psi) \end{array}}{\Diamond\phi \vee \Diamond\psi}\ Conv\,\Diamond ⩖\vee$$

$\Box$ ⩖ $\vee$-conversion

$$\dfrac{\begin{array}{c} D \\ \Box(\phi ⩖ \psi) \end{array}}{\Box\phi \vee \Box\psi}\ Conv\,\Box ⩖\vee$$

Completeness

Lemma:

$$\phi \in BSML^{\oslash} \implies \forall k \geq \mathrm{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\theta_s^k \quad \text{or} \quad \phi \dashv\vdash (\bigvee_{(M,s)\in P} \oslash\theta_s^k) \wedge \mathrm{NE}$$

$$\phi \vDash \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\theta_s^k \vDash \bigvee_{(N,t)\in Q} \oslash\theta_t^k$$

$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad s \leftrightarrows_k \biguplus R$$

$$\theta_s^k \vdash \bigvee_{(N,t)\in R} \oslash\theta_t^k$$

$$\theta_s^k \vdash \bigvee_{(N,t)\in Q} \oslash\theta_t^k$$

$$\oslash\theta_s^k \vdash \bigvee_{(N,t)\in Q} \oslash\theta_t^k$$

$$\bigvee_{(N,t)\in Q} \oslash\theta_t^k \equiv \mathop{\bigvee\!\!\!\!\bigvee}_{R\subseteq Q} \theta^k_{\biguplus R}$$

$$\implies \bigvee_{(M,s)\in P} \oslash\theta_s^k \vdash \bigvee_{(N,t)\in Q} \oslash\theta_t^k \quad \implies \quad \phi \vdash \psi$$

System for *BSML*

Omit $\mathbin{\rotatebox[origin=c]{180}{$\wedge$}\!\!\!\!\rotatebox[origin=c]{180}{$\wedge$}}$-rules and add:

$$\models \bot \mathbin{\text{W}} \mathrm{NE}$$

BSML | $BSML^{\text{W}}$

$\bot\mathrm{NE}$-translation

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi \end{matrix}}{\chi}\ \bot\mathrm{NE}\,Trs(*)$$

$(*)$ $[\psi, m]$ is W-distributive in ϕ.

W-elimination

$$\dfrac{\begin{matrix} D \\ \phi \mathbin{\text{W}} \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix}}{\chi}\ \text{W}E$$

BSML

$\Diamond\bot\mathrm{NE}$-translation

$$\dfrac{\begin{matrix} D \\ \Diamond\phi \end{matrix} \quad \begin{matrix} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi_1 \end{matrix} \quad \begin{matrix} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi_2 \end{matrix}}{\Diamond\chi_1 \vee \Diamond\chi_2} \; \Diamond\bot\mathrm{NE}\,Trs(*)$$

$\Box\bot\mathrm{NE}$-translation

$$\dfrac{\begin{matrix} D \\ \Box\phi \end{matrix} \quad \begin{matrix} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi_1 \end{matrix} \quad \begin{matrix} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi_2 \end{matrix}}{\Box\chi_1 \vee \Box\chi_2} \; \Box\bot\mathrm{NE}\,Trs(*)$$

$(*)$ $[\psi, m]$ is $\mathbin{⩔}$-distributive in ϕ.

D_1, D_2 do not contain undischarged assumptions.

$BSML^{⩔}$

$\Diamond \mathbin{⩔} \vee$-conversion

$$\dfrac{\begin{matrix} D \\ \Diamond(\phi \mathbin{⩔} \psi) \end{matrix}}{\Diamond\phi \vee \Diamond\psi} \; Conv\,\Diamond \mathbin{⩔}\vee$$

$\Box \mathbin{⩔} \vee$-conversion

$$\dfrac{\begin{matrix} D \\ \Box(\phi \mathbin{⩔} \psi) \end{matrix}}{\Box\phi \vee \Box\psi} \; Conv\,\Box \mathbin{⩔}\vee$$

Completeness

Idea: we simulate the $BSML^{\mathbb{W}}$-disjunctive normal forms using "realizations".

$$BSML^{\mathbb{W}}: \qquad \phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash \bigvee\!\!\!\!\bigvee_{(M,s)\in||\phi||} \theta_s^{md(\phi)}$$

Completeness

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$BSML$: Each ϕ is provably equivalent to some ψ of the form $[a_1, m_1] \bigcirc_1 [a_2, m_2] \bigcirc_2 \ldots \bigcirc_{n-1} [a_n, m_n]$ where $a_i \in ML \cup \{\mathrm{NE}\}$ and $\bigcirc_i \in \{\wedge, \vee\}$—i.e. ψ can constructed using $\mathbb{W}$-distributive occurrences of classical formulas and NE:

$$\phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash p \vee (\alpha \wedge \mathrm{NE}) = \psi$$

Completeness

Idea: we simulate the $BSML^{\mathbb{W}}$-disjunctive normal forms using "realizations".

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BSML: Each ϕ is provably equivalent to some ψ of the form $[a_1, m_1] \circ_1 [a_2, m_2] \circ_2 \dots \circ_{n-1} [a_n, m_n]$ where $a_i \in ML \cup \{\mathrm{NE}\}$ and $\circ_i \in \{\wedge, \vee\}$—i.e. ψ can constructed using $\mathbb{W}$-distributive occurrences of classical formulas and NE:

$$\phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash p \vee (\alpha \wedge \mathrm{NE}) = \psi$$

Replace each $[a_i, m_i]$ by some $\theta_{s_{a_i}}^{md(a_i)}$ such that $s_{a_i} \vDash a_i$. The result is a realization ψ^f of ψ:

$$\psi = p \vee (\alpha \wedge \mathrm{NE}) \qquad\qquad \psi^f = \theta_{s_p}^0 \vee (\theta_{s_\alpha}^1 \wedge \theta_{s_{\mathrm{NE}}}^0)$$

Completeness

Idea: we simulate the $BSML^{\mathbb{W}}$-disjunctive normal forms using "realizations".

$$BSML^{\mathbb{W}}: \qquad \phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash \bigvee_{(M,s)\in\|\phi\|} \theta_s^{md(\phi)}$$

$BSML$: Each ϕ is provably equivalent to some ψ of the form $[a_1, m_1] \circ_1 [a_2, m_2] \circ_2 \ldots \circ_{n-1} [a_n, m_n]$ where $a_i \in ML \cup \{\mathrm{NE}\}$ and $\circ_i \in \{\wedge, \vee\}$—i.e. ψ can constructed using $\mathbb{W}$-distributive occurrences of classical formulas and NE:

$$\phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash p \vee (\alpha \wedge \mathrm{NE}) = \psi$$

Replace each $[a_i, m_i]$ by some $\theta_{s_{a_i}}^{md(a_i)}$ such that $s_{a_i} \models a_i$. The result is a realization ψ^f of ψ:

$$\psi = p \vee (\alpha \wedge \mathrm{NE}) \qquad\qquad \psi^f = \theta^0_{s_p} \vee (\theta^1_{s_\alpha} \wedge \theta^0_{s_{\mathrm{NE}}})$$

(i) Given $a_i \equiv \mathbb{W}_{(M,s)\in\|a_i\|} \theta_s^{md(a_i)}$, and $\mathbb{W}$-distributivity:

$$\psi \equiv \mathbb{W} F_\psi = \mathbb{W}\{\psi^f \mid \psi^f \text{ is a realization for } \psi\}$$
$$\forall \psi^f \in F_\psi : \psi^f \vdash \psi$$
$$\text{if } \forall \psi^f \in F_\psi : \Gamma, \psi^f \vdash \chi, \text{ then } \Gamma, \psi \vdash \chi$$

Completeness

Idea: we simulate the $BSML^{\mathbb{W}}$-disjunctive normal forms using "realizations".

$$BSML^{\mathbb{W}}: \qquad \phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash \bigvee_{(M,s)\in\|\phi\|} \theta_s^{md(\phi)}$$

$BSML$: Each ϕ is provably equivalent to some ψ of the form $[a_1, m_1] \circ_1 [a_2, m_2] \circ_2 \ldots \circ_{n-1} [a_n, m_n]$ where $a_i \in ML \cup \{\mathrm{NE}\}$ and $\circ_i \in \{\wedge, \vee\}$—i.e. ψ can constructed using $\mathbb{W}$-distributive occurrences of classical formulas and NE:

$$\phi = p \vee (\Diamond((q \wedge \mathrm{NE}) \vee (r \wedge \mathrm{NE})) \wedge \mathrm{NE}) \dashv\vdash p \vee (\alpha \wedge \mathrm{NE}) = \psi$$

Replace each $[a_i, m_i]$ by some $\theta_{s_{a_i}}^{md(a_i)}$ such that $s_{a_i} \vDash a_i$. The result is a realization ψ^f of ψ:

$$\psi = p \vee (\alpha \wedge \mathrm{NE}) \qquad\qquad \psi^f = \theta_{s_p}^0 \vee (\theta_{s_\alpha}^1 \wedge \theta_{s_{\mathrm{NE}}}^0)$$

(i) Given $a_i \equiv \mathbb{W}_{(M,s)\in\|a_i\|} \theta_s^{md(a_i)}$, and $\mathbb{W}$-distributivity:

$$\psi \equiv \mathbb{W} F_\psi = \mathbb{W}\{\psi^f \mid \psi^f \text{ is a realization for } \psi\}$$
$$\forall \psi^f \in F_\psi : \psi^f \vdash \psi$$
$$\text{if } \forall \psi^f \in F_\psi : \Gamma, \psi^f \vdash \chi, \text{ then } \Gamma, \psi \vdash \chi$$

(ii) For each ψ^f there is some $\theta_s^{md(\psi)}$ such that $\psi^f \dashv\vdash \theta_s^{md(\psi)}$

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