Bicompleteness Theorems for Team Logics with the Dual Negation

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Helsinki Logic Seminar

Overview

Burgess [Bur03] showed that the dual/game-theoretical negation of independence-friendly logic (IF) [HS89; Hin96] and dependence logic (D) [Vä07] exhibits an extreme degree of semantic indeterminacy. We show analogues of this result for different logics.

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3. Some remarks on interpretations of the theorem

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(1)-(3) are equivalent to the existential second-order (ESO) sentence $\exists f \exists g \forall x \forall u \phi(x, f(x), u, g(u))$.
(Cf. the FO-sentence $\forall x \exists y \forall u \exists v \phi(x, y, u, v)$).

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(Cf. the FO-sentence $\forall x \exists y \forall u \exists v \phi(x, y, u, v)$).
On the level of sentences, each of IF, D, and the prenex fragment of H (H_p) is equivalent to ESO [End70; Wal70; Hin96; Vä07] $\implies$ these logics are not closed under classical negation.

Game-Theoretical Semantics for FO and IF

The motivation for adopting the dual negation in IF comes from its original semantics, which were game-theoretical.

Informal game-theoretical semantics for FO

Two players: I and II. In the game $G(M, s, \phi, i)$ ($i \in \{I, II\}$), i tries to verify the formula ϕ given model M and assignment s; the other player tries to falsify it. Rules:

- ϕ is atomic: The verifier wins $G(M, s, \phi, i)$ if $M \vDash_s \phi$. The falsifier wins if $M \nvDash_s \phi$
- ϕ is $\psi \vee \chi$: The verifier picks $\theta := \psi$ or $\theta := \chi$. Now play the game $G(M, s, \theta, i)$.
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We define $M \vDash_s \phi :\iff i$ has a winning strategy in the game $G(M, s, \phi, i)$. By the Gale-Stewart Theorem, one of the players always has a winning strategy, so $M \nvDash_s \phi \iff M \vDash_s \neg\phi$.

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This is equivalent to the standard Tarskian satisfaction definition. In particular, the game-theoretical negation of FO is just the classical negation of FO.

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The game is one of imperfect information. The Gale-Stewart Theorem does not apply: there are games in which neither player has a winning strategy.

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Matching pennies game in IF [MSS11]

Each player has a coin that they secretly turn to heads or tails. The coins are revealed simultaneously. *I* wins if the coins are both heads or both tails. Otherwise *II* wins.

We can model this using the game $G(M, \varnothing, \phi, I)$ where ϕ is the IF-sentence

$$\forall x(\exists y/\{x\})x = y$$

and M is a model with domain $\{h, t\}$. Neither player has a winning strategy, so $M \not\models \phi$ and $M \not\models \neg\phi$:

II does not have a winning strategy: all they can do is choose the value of x. If they choose h, they cannot guarantee that *I* will not choose h.

I does not have a winning strategy: *II* first chooses the value of x. *I* does not know which value was picked, so they cannot ensure that they pick the same value for y.

The Dual Negation

Another way to formulate the semantics of the negation in IF: define the semantics for negation-free formulas and for $\neg\phi$ where ϕ is atomic as usual. Then let:

$$\neg(\phi \vee \psi) := \neg\phi \wedge \neg\psi$$
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$$\neg\exists x\phi := \forall x\neg\phi$$
$$\neg\forall x\phi := \exists x\neg\phi$$
$$\neg(\exists x/W)\phi := (\forall x/W)\neg\phi$$
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Hence the name ‘dual negation’.

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Furthermore, preservation of equivalence under replacement fails: e.g.,

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Put another way, let $\|\phi\| := \{M \mid M \models \phi\}$. We may think of $\|\phi\|$ as the meaning of ϕ. Then the meaning of ϕ does not determine the meaning of $\neg\phi$:

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We do know that ϕ and $\neg\phi$ are incompatible in that it is never the case that $M \models \phi$ and $M \models \neg\phi$ (i.e., $\|\phi\| \cap \|\neg\phi\| = \varnothing$).

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Intuitively, Burgess' Theorem says that this failure of determinacy is extreme in the sense that this incompatibility is the *only* constraint that $\|\phi\|$ places on $\|\neg\phi\|$.

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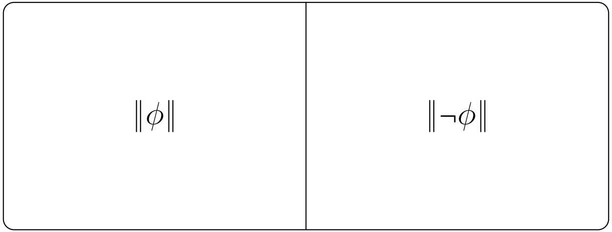
In classical logic, $M \models \neg\phi \iff M \not\models \phi$, so given $\|\phi\|$, to find $\|\neg\phi\|$, simply take the complement of $\|\phi\|$.



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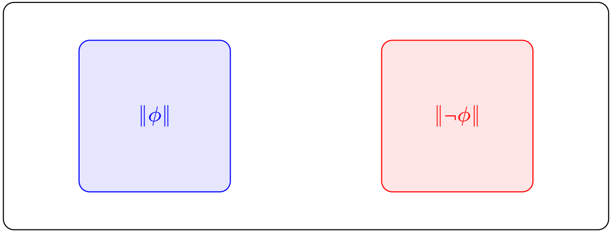

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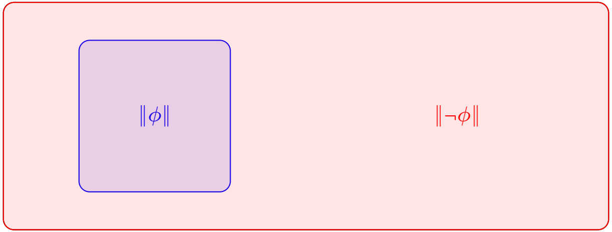

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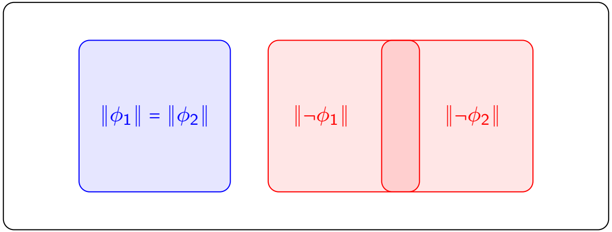

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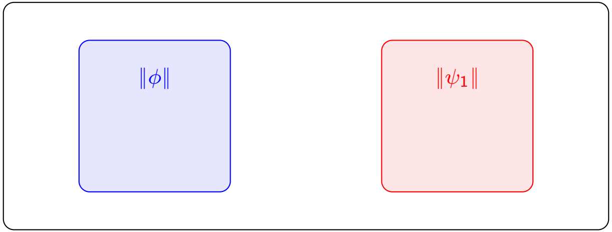

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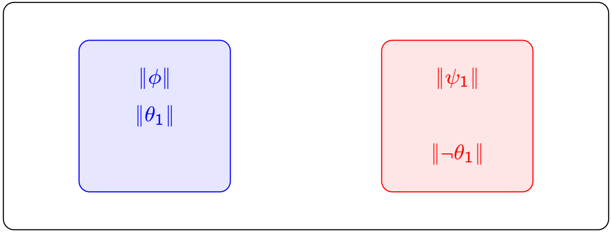

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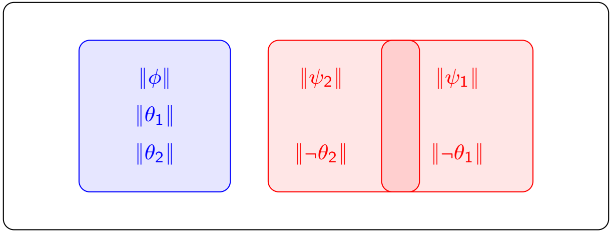
Burgess: for any IF-sentences ϕ and ψ, if $\|\phi\|$ and $\|\psi\|$ are disjoint, there is an IF-sentence θ with $\|\theta\| = \|\phi\|$ and $\|\neg\theta\| = \|\psi\|$.



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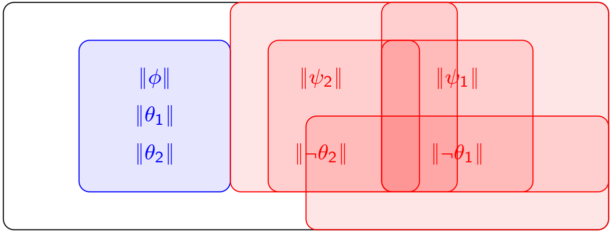

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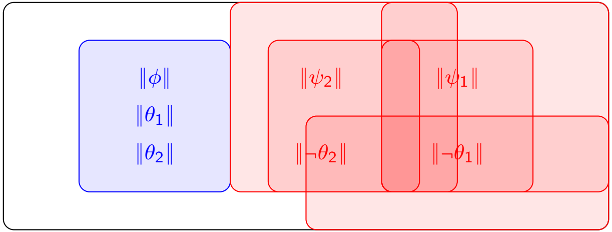

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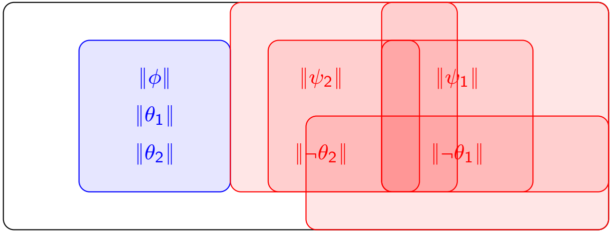

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E.g., we can have a sentence ϕ with $\phi \equiv$ "Jouko is in Helsinki" and $\neg\phi \equiv$ "Jouko is in London drinking tea, and it is Tuesday."

We assume $|dom(M)| \geq 2$ for all models M.

Lemma: There is θ_0 s.t. $\theta_0 \equiv \bot \equiv \neg\theta_0$. (E.g. the matching pennies sentence $\forall x(\exists y/\{x\})x = y$.)

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For any IF-sentences ϕ, ψ, 1 implies 2:

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Given that for each IF-sentence ϕ, it cannot be the case that $M \vDash \phi$ and $M \vDash \neg\phi$, the converse of Burgess' Theorem also holds. This means that we can think of the theorem as an expressive completeness theorem w.r.t. classes/properties of pairs (of classes of models).

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Bicompleteness Theorem for IF

IF is bicomplete for $\{(P, Q) \mid P, Q \in \|ESO\|, P \cap Q = \varnothing\}$, and hence `bicomplete` for disjoint pairs.

Proof.

$\|IF\|^{\pm,\neg} \subseteq \{(P, Q) \mid P, Q \in \|ESO\|, P \cap Q = \varnothing\}$ by $\|IF\| \subseteq \|ESO\|$ and the converse of Burgess' theorem.
$\|IF\|^{\pm,\neg} \supseteq \{(P, Q) \mid P, Q \in \|ESO\|, P \cap Q = \varnothing\}$ by $\|IF\| \supseteq \|ESO\|$ and Burgess' theorem. □

Burgess/Bicompleteness Theorem for Formulas

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Assume we have defined notions $\|\phi\|$, pair properties, etc. appropriate for formulas. E.g. $\|\phi\| = \{(M, X) \mid M \vDash_X \phi\}$.

Say that a pair (P, Q) of team properties (class of model-team pairs) is $\bot$-incompatible if $(M, X) \in P \cap Q \implies X = \varnothing$.

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We have, by [KV11] and Kontinen and Väänänen's expressivity result for D-formulas [KV09]:

Burgess & Bicompleteness Theorem for D-formulas

$\|D\|^{\pm,\neg} = \{(P, Q) \mid P, Q$ expressible in a specific way by a downward-closed ESO-sentence with the empty team property; and P and Q are ⊥-incompatible$\}$.

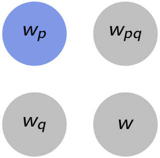
D is `bicomplete` for ⊥-incompatible pairs.

Burgess/Bicompleteness Theorems for Propositional and Modal Team Logics

In propositional/modal team semantics, formulas are evaluated w.r.t. propositional/modal teams: sets of valuations/possible worlds:

single-world semantics

$M, w \models \phi$



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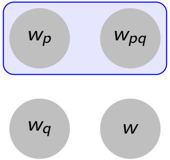

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In the propositional setting, for X is a finite set of propositional variables, P a team property (class of teams), and $\mathbb{P}$ a pair property (class of pairs of team properties), we let $\|\phi\| := \{t \mid t \vDash \phi\}$; $\|\phi\|_X := \{t \subseteq 2^X \mid t \vDash \phi\}$; $\|L\| := \{\|\phi\| \mid \phi \in L\}$; $P_X := \{Q \subseteq \wp(2^X) \mid Q \in P\}$; $\|\phi\|^{\pm,\neg} := (\|\phi\|, \|\neg\phi\|)$; $\|\phi\|_X^{\pm,\neg} := (\|\phi\|_X, \|\neg\phi\|_X)$; $\|L\|^{\pm,\neg} := \{\|\phi\|^{\pm,\neg} \mid \phi \in L\}$; $\mathbb{P}_X := \{(P, Q) \in \mathbb{P} \mid P, Q \subseteq \wp(2^X)\}$.

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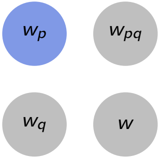

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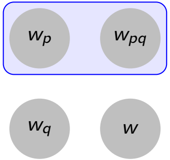

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L is expressively complete for P if $\|L\|_X = P_X$ for all finite X.
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Bilateral State-Based Modal Logic

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Teams represent speakers' information states. BSML has a bilateral semantics with two primitive semantic relations, support $\models$ and anti-support $\Dashv$. As with many other bilateral systems, the bilateralism in BSML is linked with the view that both assertion and rejection conditions must figure in meanings:

$s \models \phi$	represents	"ϕ is assertable in state s"
$s \Dashv \phi$	represents	"ϕ is rejectable in state s"

Bilateral State-Based Modal Logic

Aloni (2022) introduces a modal logic employing team semantics, Bilateral State-based Modal Logic to account for free choice inferences and related phenomena.

Free choice:	You may have coffee or tea.
	$\rightsquigarrow$ You may have coffee and you may have tea.

Teams represent speakers' information states. BSML has a bilateral semantics with two primitive semantic relations, support $\models$ and anti-support $\dashv$. As with many other bilateral systems, the bilateralism in BSML is linked with the view that both assertion and rejection conditions must figure in meanings:

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The bilateral semantics are used to define a bilateral negation:

$$s \vDash \neg\phi \qquad \text{iff} \qquad s \Dashv \phi$$

The negation is designed to ensure one also gets predictions such as the following:

Dual prohibition:	You are not allowed to eat the cake or the ice cream.
	$\rightsquigarrow$ You are not allowed to eat either one.

Syntax and semantics:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Diamond \phi \mid \mathrm{NE}$$

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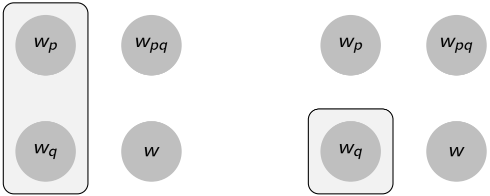
$$
\begin{array}{lll}
s \models p & \Longleftrightarrow & \forall w \in s : w \in V(p) \\
s \dashv p & \Longleftrightarrow & \forall w \in s : w \notin V(p) \\
\\
s \models \bot & \Longleftrightarrow & s = \varnothing \\
s \dashv \bot & & \text{always} \\
\\
s \models \neg\phi & \Longleftrightarrow & s \dashv \phi \\
s \dashv \neg\phi & \Longleftrightarrow & s \models \phi \\
\\
s \models \phi \vee \psi & \Longleftrightarrow & \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi \\
s \dashv \phi \vee \psi & \Longleftrightarrow & s \dashv \phi \text{ and } s \dashv \psi \\
\\
s \models \phi \wedge \psi & \Longleftrightarrow & s \models \phi \text{ and } s \models \psi \\
s \dashv \phi \wedge \psi & \Longleftrightarrow & \exists t, t' : t \cup t' = s \text{ and } t \dashv \phi \text{ and } t' \dashv \psi \\
\\
s \models \mathrm{NE} & \Longleftrightarrow & s \neq \varnothing \\
s \dashv \mathrm{NE} & \Longleftrightarrow & s = \varnothing \\
\\
s \models \Diamond\phi & \Longleftrightarrow & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi \\
s \dashv \Diamond\phi & \Longleftrightarrow & \forall w \in s : R[w] \dashv \phi
\end{array}
$$

$$R[w] = \{v \mid wRv\}$$

Split disjunction $\vee$

$$s \models \phi \vee \psi \iff \exists t, t' : \quad t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi$$

$$s \mathrel{=\!\!|} \phi \vee \psi \iff s \mathrel{=\!\!|} \phi \text{ and } s \mathrel{=\!\!|} \psi$$



(a) $s \models p \vee q$ (b) $s \models p \vee q$

The non-emptiness atom NE

$$s \models \mathrm{NE} \iff s \neq \varnothing$$
$$s \dashv \mathrm{NE} \iff s = \varnothing$$

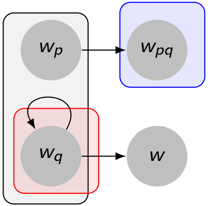
(a) $s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

(b) $s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

The modality $\Diamond$

$$R[w] = \{v \in W \mid wRv\}$$

$$s \vDash \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi$$



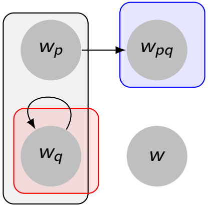
$s \vDash \Diamond q$ since

$\{w_q\} \subseteq R[w_q]$, $\{w_q\} \vDash q$ and $\{w_{pq}\} \subseteq R[w_p]$, $\{w_{pq}\} \vDash q$

The modality $\Box := \neg \Diamond \neg$

$$R[w] = \{v \in W \mid wRv\}$$

$$s \vDash \Box\phi \quad \Longleftrightarrow \quad \forall w \in s : R[w] \vDash \phi$$



$s \vDash \Box q$ since

$\{w_q\} = R[w_q]$
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and

$\{w_{pq}\} = R[w_p]$
$\{w_{pq}\} \vDash q$

Closure properties

Definition

ϕ is *downward closed*:	$[s \models \phi$ and $t \subseteq s] \implies t \models \phi$
ϕ is *union closed*:	$[s \models \phi$ for all $s \in S \neq \varnothing] \implies \bigcup S \models \phi$
ϕ has the *empty team property*:	$\varnothing \models \phi$
ϕ is *flat*:	$s \models \phi \iff \{v\} \models \phi$ for all $v \in s$
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flat $\iff$ downward closed & union closed & empty team property

Formulas of classical modal logic *ML* (the NE-free fragment of *BSML*) are flat and their team semantics coincide with their standard semantics on singletons:

$$\text{for } \alpha \in ML: \quad s \models \alpha \iff \forall v \in s : \{v\} \models \alpha \iff \forall v \in s : v \models \alpha$$

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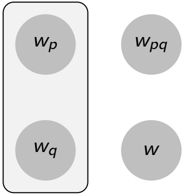
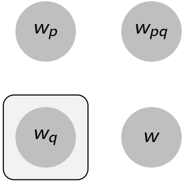
Therefore, *BSML* is an extension of classical modal logic (K):

$$\text{for } \Xi \cup \{\alpha\} \subseteq ML: \quad \Xi \models \alpha \text{ (in team semantics)} \iff \Xi \models \alpha \text{ (in standard semantics)}$$

Downward closure and the empty team property fail in *BSML* due to NE:



(a) $s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$



(b) $s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

However, formulas of *BSML* are union closed and convex, and furthermore:

Expressive Completeness Theorem [AK25]

BSML is expressively complete for convex union-closed (and modally definable) modal team properties:

$$\|BSML\|_X = \{P \mid P \text{ union closed, convex, and invariant under bounded team bisimulation}\}_X.$$

for all finite X.

The following dual equivalences hold for the negation (where $\Box := \neg \Diamond \neg$):

$$\neg\neg\phi \equiv \phi \qquad \neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$

$$\neg \mathrm{NE} \equiv \bot \qquad \neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$$

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We define the following abbreviation:

Strong contradiction $\perp\!\!\!\perp := \bot \wedge \mathrm{NE}$. $s \models \perp\!\!\!\perp$ is never the case.

(Strong) tautology $\top := \neg\bot$. $s \models \top$ is always the case.

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- Strong contradiction $⫫ := \bot \wedge \mathrm{NE}$. $s \models ⫫$ is never the case.
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As with IF and D, replacement of equivalents does not hold under negation:

$$\neg\mathrm{NE} \equiv \bot \text{ but } \neg\neg\mathrm{NE} \equiv \mathrm{NE} \not\equiv \top \equiv \neg\bot.$$

Adapting Burgess' Theorem for BSML

Kontinen and Väänänen's Burgess theorem for D employs the following notion of incompatibility (reformulated for the propositional/modal setting):

ϕ and ψ are ⊥-incompatible : $\iff$ $\phi, \psi \models \bot$

$t \models \phi$ and $t \models \psi$ implies $t = \emptyset$

[1] Strictly speaking in the modal setting, $|\phi| = \{(M, w) \mid \exists (M, s) \in \|\phi\| : w \in s\}$.

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$$P \text{ and } Q \text{ are ground-incompatible}: \quad \bigcup P \cap \bigcup Q = \varnothing$$

Lemma: For $\phi \in BSML$, ϕ and $\neg\phi$ are ground-incompatible.

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Therefore, by Lemma, $\{w_p\} \cap \{w_p, w_{\neg p}\} = \{w_p\} = \varnothing$, a contradiction.

[1] Strictly speaking in the modal setting, $|\phi| = \{(M, w) \mid \exists (M, s) \in \|\phi\| : w \in s\}$.

We have already noted that $\|BSML\| \subseteq \{(P, Q) \mid P, Q$ ground-incompatible$\}$—the converse of a ground-incompatibility Burgess theorem for *BSML*.

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We now further show the Burgess and bicompleteness theorems:

Burgess/Bicompleteness Theorem for *BSML*

The following are equivalent:

1. $|\phi| \cap |\psi| = \varnothing$ (i.e., $[s \vDash \phi$ and $t \vDash \psi] \implies s \cap t = \varnothing$)
2. There is a θ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Therefore,

$$\|BSML\|_X^{\pm,\neg} = \{(P,Q) \mid P, Q \text{ union closed, convex and invariant under bounded bisimulation; } P \text{ and } Q \text{ ground-incompatible}\}_X$$

for all finite X.
So *BSML* is `bicomplete` for ground-incompatible pairs.

Lemma 1: There is θ_0 s.t. $\theta_0 \equiv \bot \equiv \neg\theta_0$.
Let $\theta_0 := \Diamond(\bot\!\!\bot \vee \neg \bot\!\!\bot)$. Then:

$$\Diamond(\bot\!\!\bot \vee \neg \bot\!\!\bot) \equiv \Diamond \bot\!\!\bot \equiv \bot$$

$$\neg\Diamond(\bot\!\!\bot \vee \neg \bot\!\!\bot) \equiv \Box\neg(\bot\!\!\bot \vee \neg \bot\!\!\bot) \equiv \Box(\neg \bot\!\!\bot \wedge \bot\!\!\bot) \equiv \Box \bot\!\!\bot \equiv \bot$$

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$$\Diamond(\amalg \vee \neg \amalg) \equiv \Diamond \amalg \equiv \bot$$

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Separation Theorem: If $|\phi| \cap |\psi| = \varnothing$, then there is an η s.t. $\phi \models \eta$ and $\psi \models \neg\eta$.
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(Define ϕ' by putting ϕ in negation normal form and replacing each $\neg\mathrm{NE}$ by $\bot$.)

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Burgess/Bicompleteness Theorem for *BSML*

The following are equivalent:

1. $|\phi| \cap |\psi| = \varnothing$ (i.e., $[s \vDash \phi$ and $t \vDash \psi] \implies s \cap t = \varnothing$)
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Proof.

$1 \implies 2$: Let $\phi_0 := \phi' \vee \theta_0$ and $\psi_0 := \psi' \vee \theta_0$ with θ_0 from Lemma 1 and ϕ', ψ' from Lemma 2. Then:

$$\begin{array}{ccccccccc} \phi_0 & \equiv & \phi' \vee \theta_0 & \equiv & \phi' \vee \bot & \equiv & \phi & & \\ \neg\phi_0 & \equiv & \neg(\phi' \vee \theta_0) & \equiv & \neg\phi' \wedge \neg\theta_0 & \equiv & \neg\phi' \wedge \bot & \equiv & \bot \end{array}$$

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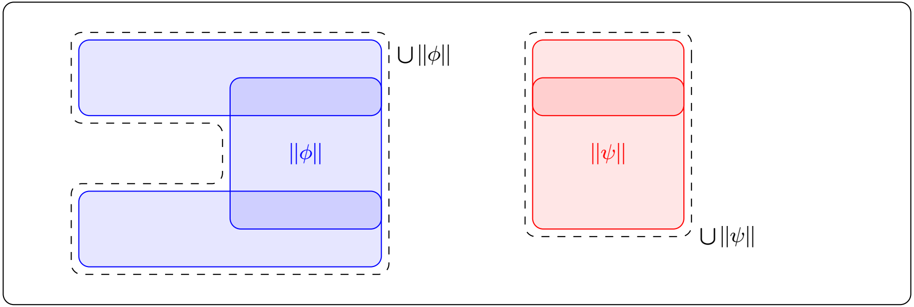
Let $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$. Then:

$$\begin{array}{lllllll} \theta & \equiv \phi_0 \wedge (\neg\psi_0 \vee \eta) & \equiv \phi_0 \wedge (\bot \vee \eta) & \equiv \phi_0 \wedge \eta & \equiv \phi_0 & \equiv \phi & \\ \neg\theta & \equiv \neg(\phi_0 \wedge (\neg\psi_0 \vee \eta)) & \equiv \neg\phi_0 \vee \neg(\neg\psi_0 \vee \eta) & \equiv \bot \vee (\neg\neg\psi_0 \wedge \neg\eta) & \equiv \psi_0 \wedge \neg\eta & \equiv \psi_0 & \equiv \psi \quad \square \end{array}$$

ϕ, ψ are ground-incompatible if $[t \vDash \phi$ and $s \vDash \psi] \implies s \cap t = \varnothing$. Equivalently, $|\phi| \cap |\psi| = \bigcup \|\phi\| \cap \bigcup \|\psi\| = \varnothing$.)



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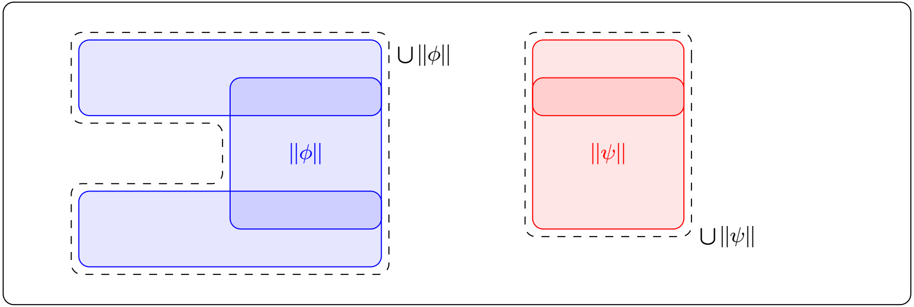

Relationship between ground-incompatibility and $\perp$-incompatibility

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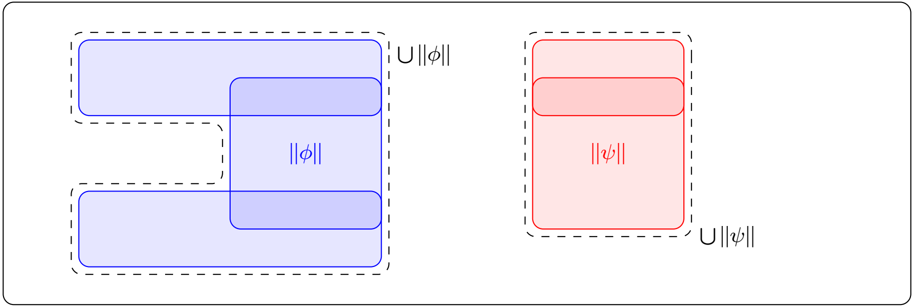

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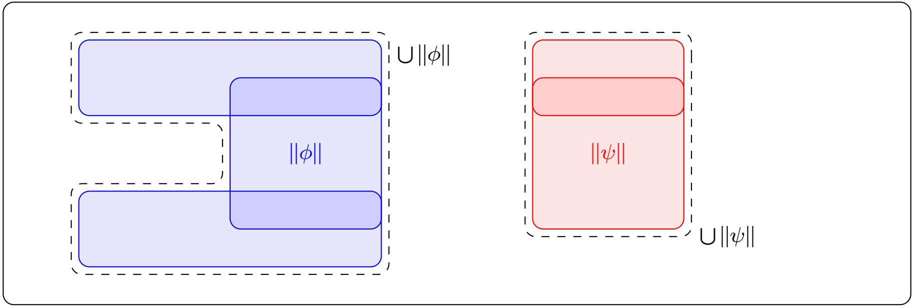

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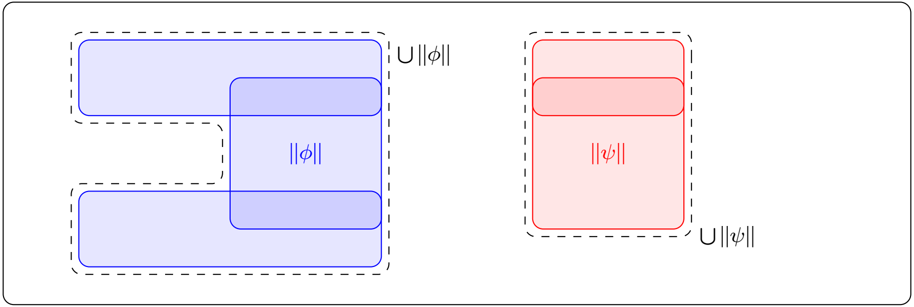

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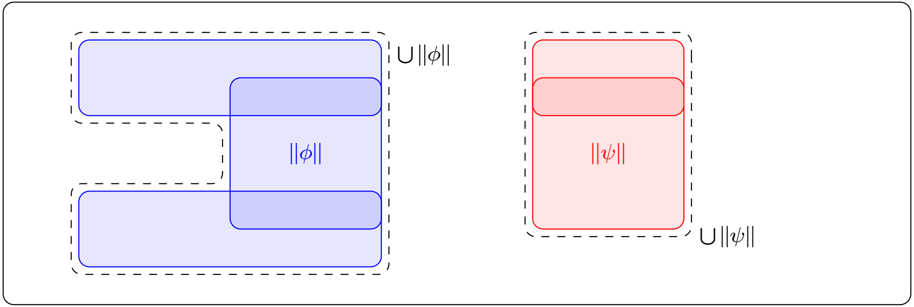

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Therefore, D is also `bicomplete` for ground-incompatible pairs.

Sketch of one possible intuitive interpretation of ground-incompatibility (cf. [Yal07; Yal11; CGR18]):

Recall that teams can represent information states.

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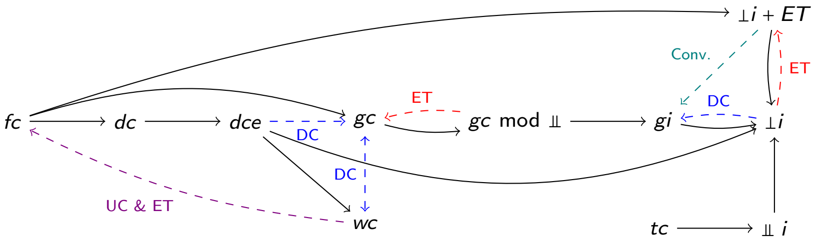
Cf. epistemic contradictions: It is raining but it might not be raining.
$\blacklozenge r \wedge \neg r \vDash \bot$ but $|\blacklozenge r| \cap |\neg r| = |\top| \cap |\neg r| = |\neg r| \neq \varnothing$.

More incompatibility notions/pair properties and `bicompleteness` results from the paper:

Pair property	Definition(s)	Bicomplete logics
$\bot$-incompatible ($\bot i$)	• $[s \models \phi_0$ and $s \models \phi_1] \implies s = \varnothing$ • $\phi_0, \phi_1 \models \bot$ • $\bot i + ET$ or $⫫\ i$	D, IF
Ground-incompatible (gi)	• $[s \models \phi_0$ and $t \models \phi_1] \implies s \cap t = \varnothing$ • $\lvert\phi_0\rvert \cap \lvert\phi_1\rvert = \varnothing$	D, IF, $BSML$, $BSML^{\mathbb{V}}$, $PL(\text{NE}, \mathbb{V})$
$⫫$-incompatible ($⫫\ i$)	• ϕ_0 and ϕ_1 never jointly true • $\phi_0, \phi_1 \models ⫫$ • $\lVert\phi_0\rVert \cap \lVert\phi_1\rVert = \varnothing = \lVert ⫫ \rVert$	
$\bot$-incompatible and empty team prop. ($\bot i + NE$)	• $[s \models \phi_0$ and $s \models \phi_1] \iff s = \varnothing$ • $\phi_0, \phi_1 \models \bot$ and $\phi_0, \phi_1 \not\models ⫫$ • $\lVert\phi_0\rVert \cap \lVert\phi_1\rVert = \{\varnothing\} = \lVert\bot\rVert$	D, IF
World-complementary (wc)	• $\{w\} \models \phi_0 \iff \{w\} \not\models \phi_1$	$PL(=\!\!(\cdot))$, PL
Team-complementary (tc)	• $s \models \phi_i \iff s \not\models \phi_{1-i}$ • $\lVert\phi_i\rVert = \lVert\top\rVert \setminus \lVert\phi_{1-i}\rVert$	$PL(\sim)$ (w.r.t. $\sim$)
Flat-complementary (fc)	• wc and ϕ_0, ϕ_1 flat • $\lVert\phi_i\rVert = \wp(\lvert\top\rvert \setminus \lvert\phi_{1-i}\rvert)$ • $\lVert\phi_i\rVert = \{s \mid t \models \phi_{1-i} \implies s \cap t = \varnothing\}$ • $\lVert\phi_i\rVert = \bigcup\{P \subseteq \lVert\top\rVert \mid P, \lVert\phi_{1-i}\rVert$ G-I$\}$	PL
ϕ_1 down-set complement (dc) of ϕ_0	• $s \models \phi_1 \iff$ $[[t \models \phi_0$ and $t \subseteq s] \implies t = \varnothing]$	$InqB$ (w.r.t. $\neg_i$), PL
Down-set complements (on either side) (dce)	• ϕ_1 dc of ϕ_0 or ϕ_0 dc of ϕ_1	HS, PL
Ground-complementary (gc)	• $\lvert\phi_i\rvert = \lvert\top\rvert \setminus \lvert\phi_{1-i}\rvert$	$PL(=\!\!(\cdot))$, PL
Ground-complementary mod $⫫$	• $\lvert\phi_i\rvert = \lvert\top\rvert \setminus \lvert\phi_{1-i}\rvert$ or $\phi_0 \equiv ⫫$ or $\phi_1 \equiv ⫫$	$PL(\text{NE})$, $PL(=\!\!(\cdot))$, PL
All pairs		$PL(\text{NE}^*, \mathbb{V})$

More relationships between the incompatibility notions/pair properties:



Propositional Dependence Logic with the Dual Negation

Propositional dependence logic $PL(=(\cdot))$, similarly to D, extends classical propositional logic PL with dependence atoms:

Syntax of $PL(=(\cdot))$:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \; =(p_1, \ldots, p_n, q)$$

Bilateral semantics for dependence atoms:

$$s \models =(p_1 \ldots, p_n, q) \quad :\Longleftrightarrow \quad \forall v, w \in s : [v \models p_i \iff w \models p_i \text{ for all } \forall 1 \leq i \leq n] \implies [v \models q \iff w \models q]$$

$$s \dashv =(p_1 \ldots, p_n, q) \quad :\Longleftrightarrow \quad s = \varnothing$$

In other words, a dependence atom $=(p_1 \ldots, p_n, q)$ is true/supported in a team s if the values of $p_i, \ldots, p_n$ jointly determine the value of q in any valuation in the team.

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$$s \dashv =(p_1 \dots, p_n, q) \quad :\iff \quad s = \emptyset$$

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Expressive Completeness Theorem [YV16]

$PL(=(\cdot))$ is expressively complete for downward-closed properties with the empty team property.

We show that $PL(=(\cdot))$ is `bicomplete` for ground-complementary pairs:

$$\phi \text{ and } \psi \text{ are ground-complementary}: \qquad |\phi| = |\top| \setminus |\psi|$$
$$P \text{ and } Q \text{ are ground-complementary}: \qquad \bigcup P = |\top| \setminus \bigcup Q$$

(Intuitively, ϕ and ψ are ground-complementary if the factual information expressed by one is the classical negation of the factual information expressed by the other.)

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We first show that ϕ and $\neg\phi$ are ground-complementary.

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Lemma

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Proof.

By induction on the complexity of ϕ.

Define the flattening ϕ^f of ϕ by $\phi^f := \phi(\top/ =(p_1 \ldots, p_n, q))$ (i.e., replace each dependence atom by $\top$). Clearly $\phi^f \in PL$.

For $\alpha \in PL$, define $[\![\alpha]\!] := \{w \mid w \vDash \alpha\}$. Clearly $[\![\alpha]\!] = |\alpha|$ for $\alpha \in PL$.

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For all $\phi \in PL(=(\cdot))$, $|\phi| = |\phi^f|$.

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Base: $| =(p_1 \ldots, p_n, q)| = |\top| = | =(p_1 \ldots, p_n, q)^f|$ since $w \in \{w\} \vDash =(p_1 \ldots, p_n, q)$ for any w, and $|\neg =(p_1 \ldots, p_n, q)| = |\bot| = |(\neg =(p_1 \ldots, p_n, q))^f|$.

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Let $\phi = \psi \wedge \chi$. We first show $|\psi \wedge \chi| = |\psi| \cap |\chi|$. $|\psi \wedge \chi| \subseteq |\psi| \cap |\chi|$ is immediate; for the converse inclusion, let $w \in |\psi| \cap |\chi|$. Then $w \in t \vDash \psi$ and $w \in u \vDash \chi$. By downward closure, $\{w\} \vDash \psi$ and $\{w\} \vDash \chi$, so $w \in |\psi \wedge \chi|$. Then:

$$|\psi \wedge \chi| = |\psi| \cap |\chi| = |\psi^f| \cap |\chi^f| = [\![\psi^f]\!] \cap [\![\chi^f]\!] = [\![\psi^f \wedge \chi^f]\!] = [\![(\psi \wedge \chi)^f]\!] = |(\psi \wedge \chi)^f|.$$

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Cf. first-order dependence logic (this is essentially D-version of the matching pennies sentence):

$$|\forall x =\!(x)| = |\bot| \text{ (assuming models of size } \geq 2); \text{ and } |\neg\forall x =\!(x)| = |\bot|$$

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Characteristic formulas for valuations and teams:

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$\gamma_0^X := \bot$, $\gamma_1^X := \bigwedge_{p \in X} {=}(p)$, and for $n \geq 2$, $\gamma_n^X := \bigvee_n \gamma_1$.
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$$(P, Q) = \left\| \bigwedge_{s \in \|\top\|_X \setminus P} \xi_s^X \vee \neg \bigwedge_{s \in (\|\top\|_X \setminus Q)^{>1}} \xi_s^X \right\|_X^{\pm} \qquad \text{where } R^{>1} := \{s \in R \mid |s| > 1\}.$$

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Note, however, that Hintikka did also consider an extension of IF with the Boolean/contradictory negation $\sim$ (*extended independence-friendly logic*), and that he ultimately viewed each negation as indispensable [Hin96]:

> *...in any sufficiently rich language, there will be two different notions of negation present. Or if you prefer a different formulation, our ordinary concept of negation is intrinsically ambiguous. The reason is that one of the central things we certainly want to express in our language is the contradictory negation. But ... a contradictory negation is not self-sufficient. In order to have actual rules for dealing with negation, one must also have the dual negation present, however implicitly.*

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Burgess' theorem is a further fact pertaining to the *degree* of failure of determination. Our different incompatibility notions can be thought of as giving us a way of measuring this degree, with `bicompleteness` w.r.t. a stronger notion of incompatibility corresponding to less failure of determination:

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One should also be careful about the claim that the negation is not a semantic operation. Hodges' [Hod97] goal in developing team semantics was to provide a compositional semantics for IF (i.e., all connectives are "semantic operations"). Take the semantic value of ϕ to be the pair $(||\phi||, ||\neg\phi||)$. Then the value for $\neg\phi$ can be obtained from the value for ϕ by simply flipping the elements of the pair (since double negation elimination holds). This also accords with the philosophical theory motivating some bilateral notions of negation, bilateralism/rejectivism [Pri83; Pri90; Smi96; Rum00], the view that both assertion and rejection conditions must be taken into account (and they might be independent) when analyzing meanings.

A common gloss has it that Burgess establishes or shows that the dual negation is "not a semantic operation".

But note that it is not Burgess' theorem that establishes the "non-semantic" nature of the dual negation. The failure of the negation to correspond to any operation on classes of models is an easily observable, simple fact (e.g. in dependence logic: $\neg =(x,y) \equiv \bot$, but $\neg\neg =(x,y) \not\equiv \neg\neg\bot$).

Burgess' theorem is a further fact pertaining to the *degree* of failure of determination. Our different incompatibility notions can be thought of as giving us a way of measuring this degree, with `bicompleteness` w.r.t. a stronger notion of incompatibility corresponding to less failure of determination:

- Given a classical notion of negation, $\|\phi\|$ determines $\|\neg\phi\|$: $\|\neg\phi\| = \|\top\| \setminus \|\phi\|$.
- In a logic `bicomplete` for ground-complementary pairs, $\|\phi\|$ does not determine $\|\neg\phi\|$, but we at least know the ground team of $|\neg\phi|$ because $|\phi|$ determines $|\neg\phi|$: $|\neg\phi| = |\top| \setminus |\phi|$.
- In a logic `bicomplete` for ground-incompatible pairs, $\|\phi\|$ does not determine $\|\neg\phi\|$ and $|\phi|$ does not determine $|\neg\phi|$, but we know $|\phi|$ and $|\neg\phi|$ are disjoint.
- In a logic `bicomplete` for $\bot$-incompatible pairs, we do not even know that, but we know $\|\phi\| \cap \|\neg\phi\| \subseteq \{\varnothing\}$.

One should also be careful about the claim that the negation is not a semantic operation. Hodges' [Hod97] goal in developing team semantics was to provide a compositional semantics for IF (i.e., all connectives are "semantic operations"). Take the semantic value of ϕ to be the pair $(||\phi||, ||\neg\phi||)$. Then the value for $\neg\phi$ can be obtained from the value for ϕ by simply flipping the elements of the pair (since double negation elimination holds). This also accords with the philosophical theory motivating some bilateral notions of negation, bilateralism/rejectivism [Pri83; Pri90; Smi96; Rum00], the view that both assertion and rejection conditions must be taken into account (and they might be independent) when analyzing meanings. More carefully, then: the dual negation is not semantic w.r.t. positive meanings, but it is semantic w.r.t. positive-negative meaning pairs.

Thank you!

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