Convex Team Logics

Aleksi Anttila & Søren Brinck Knudstorp

ILLC, University of Amsterdam

Workshop on the Occasion of Marco Degano's Doctoral Defense

Plan for the talk

- Convexity: What is it and why is it interesting?
- Team Logics: Connectives and notions of propositionhood.
- Results: Expressive completeness for convex team logics.

Convexity: the why and what

Degano, 2024: *The underlying idea is that the meaning of expressions should denote a convex 'region' provided a suitable notion of meaning space. Convexity would be violated when gaps are present in the underlying 'region' that expressions denote.*

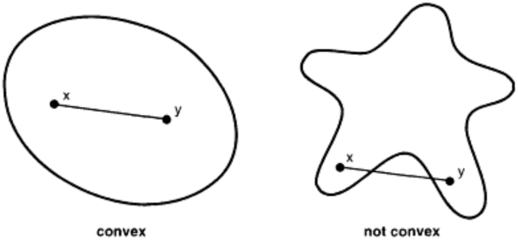
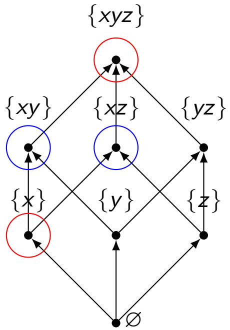


Image from Gärdenfors, The Geometry of Meaning: Semantics Based on Conceptual Spaces, 2000



Convexity as Linguistic/Cognitive Universal

1. **Generalized quantifiers:**

   Barwise & Cooper, 1981: *The simple NP's of any natural language express monotone quantifiers or conjunctions of monotone quantifiers.*

   Van Benthem, 1984: *Monotonicity is a strong condition, whose validity for arbitrary logical constants is debatable. Nevertheless, one does expect a certain "smooth" behaviour of reasonable quantifiers; and, therefore, the following notion of continuity [ed: convexity] has a certain interest...*

2. **Concept formation:**

   Gärdenfors, 2000: *A central feature of our cognitive mechanisms is that we assign properties to the objects that we observe [...] I primarily want to pin down the properties that are, in a sense, natural to our way of thinking [...] The third and most powerful criterion of a region is the following, which also relies on betweenness: A subset C of a conceptual space S is said to be convex if, for all points x and y in C, all points between x and y are also in C.*

Convexity as Linguistic/Cognitive Universal

3. **Indefinites:**

   Degano, 2024: *We can then provide a more grounded explanation for the absence of indefinites that lexicalize only the SK and NS functions as a violation of the convexity constraint.*

Definition (Convexity over Teams)

A set of teams $\mathcal{P}$ is convex iff for all t, t', t'' such that $t \subseteq t' \subseteq t''$, if $t \in \mathcal{P}$ and $t'' \in \mathcal{P}$, then $t' \in \mathcal{P}$.

(Propositional) team logics: connectives

Traditionally (in, e.g., CPC), formulas φ are evaluated at single valuations $v : \mathbf{Prop} \to \{0,1\}$,

$$v \models \varphi.$$

In team semantics, formulas φ are evaluated at sets ('teams') of valuations $t \subseteq \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$,

$$t \models \varphi.$$

Definition (some team-semantic clauses)

For $t \subseteq \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$, we define

$t \models p$	**iff**	$\forall v \in t : v(p) = 1$,
$t \models \varphi \wedge \psi$	**iff**	$t \models \varphi$ *and* $t \models \psi$,
$t \models \varphi \vee \psi$	**iff**	there exist t', t'' such that $t' \models \varphi$; $t'' \models \psi$; and $t = t' \cup t''$,
$t \models \varphi \mathbin{\vee\!\!\!\vee} \psi$	**iff**	$t \models \varphi$ *or* $t \models \psi$.

New connectives

On connectives:

Fact 1: Team semantics for $\{\neg, \wedge, \vee\}$ gives us classical logic.

Fact 2: In classical logic, $\{\neg, \wedge, \vee\}$ is famously functionally complete: all other connectives are definable by these.

Fact 3: In team semantics, $\{\neg, \wedge, \vee\}$ can only capture a fraction of the expressible connectives. For example, $\mathbin{\rotatebox[origin=c]{180}{$\mathbb{A}$}}$ is not definable using $\{\neg, \wedge, \vee\}$.

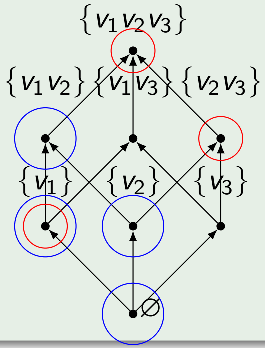
Consequence: We have a semantic framework for expressions beyond classical assertions, such as questions.

Take-away: *Teams provide for ways to express meanings not readily expressible in single-valuation semantics;* and thus for considering new connectives!

(Propositional) team logics: propositionhood

- Given any condition-based semantics, we obtain a notion of propositionhood defined as a set of conditions. *Slogan:* Proposition = a set of conditions.
- In team semantics, conditions are teams.
- So, propositions are sets of teams. **Caveat:** The standard terminology is not 'propositions' but 'properties'.

Example



Since our meaning space now has structure (as powersets), we can consider natural restrictions on what a proposition is. Or what different kinds of propostions/meanings there are! For instance, assertions contra questions. (Note the analogy with generalized quantifiers.)

Notions of propositionhood (closure properties)

Take-away: *Teams provide for ways to express meanings not readily expressible in single-valuation semantics; and thus for considering new notions of propositionhood!*

Definition (some restrictions on propositionhood)

ϕ is *downward closed*:	$[s \vDash \phi$ and $t \subseteq s] \implies t \vDash \phi$
ϕ is *union closed*:	$[s \vDash \phi$ for all $s \in S \neq \varnothing] \implies \bigcup S \vDash \phi$
ϕ has the *empty team property*:	$\varnothing \vDash \phi$
ϕ is *flat*:	$s \vDash \phi \iff \{v\} \vDash \phi$ for all $v \in s$
ϕ is *convex*:	$[s \vDash \phi, u \vDash \phi$ and $s \subseteq t \subseteq u] \implies t \vDash \phi$

Convexity generalizes downward closure:

$$\text{downward closed} \implies \text{convex}$$

Interface of connectives and propositionhood

The choice of connectives and the corresponding notion of propositionhood are closely connected. Here are some examples:

- Classical formulas are flat (so union closed) [i.e., classical assertions are flat]
- Formulas with $\mathbin{\mathbb{V}}$ might not be union closed. [i.e., questions are not union closed]
- Consider the *epistemic might* operator $\blacklozenge$, defined as

$$s \models \blacklozenge\phi \iff \exists t \subseteq s : t \neq \emptyset \;\&\; t \models \phi.$$

Formulas with $\blacklozenge$ are not downward closed [i.e., epistemic uncertainty is not persistent]

Convexity

Recall Degano, 2024: *The underlying idea is that the meaning of expressions should denote a convex 'region' provided a suitable notion of meaning space*

To summarize, we paraphrase: *The underlying idea is that* $\|\varphi\|$ *should denote a convex 'region': if* $s, u \in \|\varphi\|$ *and* $s \subseteq t \subseteq u$*, then* $t \in \|\varphi\|$

Expressive completeness

We answer an open question concerning the expressive power of a certain propositional team logic by showing it is capable of capturing the full range of convex and union-closed propositions (properties). We also find logics capable of expressing all convex propositions.

We say a logic L is *expressively complete* for a class of properties $\mathbb{P}$ ($\|L\| = \mathbb{P}$) if

(i) $\|L\| \subseteq \mathbb{P}$: each property $\|\phi\|$ (where $\phi \in L$) is in $\mathbb{P}$

(ii) $\mathbb{P} \subseteq \|L\|$: each property $\mathcal{P} \in \mathbb{P}$ can be expressed by a formula of L: $\mathcal{P} = \|\phi\|$ where $\phi \in L$.

Example: Propositional dependence logic is expressively complete for the class of downward-closed (propositional) team properties

$$\mathbb{D} = \{\mathcal{P} \mid [t \in \mathcal{P} \;\&\; s \subseteq t] \implies s \in \mathcal{P}\}$$

Propositional inquisitive logic is also expressively complete for $\mathbb{D}$.

We consider one propositional logic complete for the class of convex and union-closed (propositional) team properties

$$\mathbb{CU} = \{\mathcal{P} \mid [[s, u \in \mathcal{P} \,\&\, s \subseteq t \subseteq u] \implies t \in \mathcal{P}] \,\&\, [s, u \in \mathcal{P} \implies s \cup u \in \mathcal{P}]\}.$$

This logic is the propositional fragment of Aloni's Bilateral State-based Modal Logic.

We also consider two logics complete for the class of convex (propositional) team properties

$$\mathbb{C} = \{\mathcal{P} \mid [s, u \in \mathcal{P} \,\&\, s \subseteq t \subseteq u] \implies t \in \mathcal{P}\}.$$

These logics are (in a sense) convex variants of the downward-closed logics propositional dependence logic and propositional inquisitive logic.

A Logic for Convex Union-closed Properties

Syntax of classical propositional logic (with $\vee$) $\mathbf{PL}_\vee$

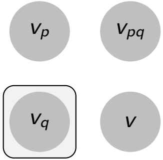
$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

We extend $\mathbf{PL}_\vee$ with the nonemptiness atom NE—syntax of $\mathbf{PL}_\vee(\mathrm{NE})$:

$$\phi ::= p \mid \bot \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \mathrm{NE}$$

where $\alpha \in \mathbf{PL}_\vee$.

$$t \vDash \mathrm{NE} \iff t \neq \varnothing$$



$\{v_q\} \vDash p \vee q$;
$\{v_q\} \not\vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

Aloni's (2022) Bilateral State-based Modal Logic is a modal extension of $\mathbf{PL}_\vee(\mathrm{NE})$ (and is similarly complete for convex union-closed modal team properties in the modal setting). Aloni uses NE to model a process of pragmatic enrichment which is then used to account for free choice inferences and other phenomena. E.g.,:

You may have coffee or tea $\rightsquigarrow$ You may have coffee and you may have tea.
$$\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \vDash \Diamond c \wedge \Diamond t$$

To show $\mathbf{PL}_\vee(\mathrm{NE}) = \mathbb{CU}$, we show:

(i) $\|\mathbf{PL}_\vee(\mathrm{NE})\| \subseteq \mathbb{CU}$: by induction.

(ii) $\mathbb{CU} \subseteq \|\mathbf{PL}_\vee(\mathrm{NE})\|$: by constructing characteristic formulas for properties in $\mathbb{CU}$.

Characteristic formulas for valuations and teams:

$$\chi_v := \bigwedge\{p \mid v \models p\} \wedge \bigwedge\{\neg p \mid v \not\models p\} \qquad \chi_s := \bigvee_{v\in s} \chi_v$$

$$w \models \chi_v \iff w = v \qquad t \models \chi_s \iff t \subseteq s$$

Characteristic formulas for flat (downward- and union-closed) properties:

$$t \models \bigvee_{s\in\mathcal{P}} \chi_s \iff t \subseteq \bigcup \mathcal{P}$$

Characteristic formulas for upward-closed properties:

$$t \models \bigwedge_{v_1\in t_1,\dots,v_n\in t_n} (((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}) \vee \top) \iff \exists s \in \mathcal{P} = \{t_1,\dots,t_n\} : s \subseteq t$$

Characteristic formulas for convex union-closed properties:

$$t \models \bigvee_{v\in s} \chi_v \wedge \bigwedge_{v_1\in t_1,\dots,v_n\in t_n} (((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}) \vee \top) \iff \exists s \in \mathcal{P} = \{t_1,\dots,t_n\} : s \subseteq t \text{ and } t \subseteq \bigcup \mathcal{P}$$

$$\iff t \in \mathcal{P} \text{ (if } \mathcal{P} \in \mathbb{CU})$$

Logics for Convex Properties

To get a characteristic formula for all convex properties, we can replace the characteristic formula for flat properties with a characteristic formula for downward-closed properties.

Flat (downward- and union-closed) properties:

$$t \models \phi^F_{\mathcal{P}} \iff t \subseteq \bigcup \mathcal{P}$$

Upward-closed properties:

$$t \models \phi^U_{\mathcal{P}} \iff \exists s \in \mathcal{P} : s \subseteq t$$

Downward-closed properties:

$$t \models \phi^D_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \subseteq s$$

Convex union-closed properties:

$$t \models \phi^F_{\mathcal{P}} \wedge \phi^U_{\mathcal{P}} \iff \exists s \in \mathcal{P} : s \subseteq t \text{ and } t \subseteq \bigcup \mathcal{P}$$
$$\iff t \in \mathcal{P} \text{ (if } \mathcal{P} \in \mathbb{CU})$$

Convex properties:

$$t \models \phi^D_{\mathcal{P}} \wedge \phi^U_{\mathcal{P}} \iff \exists s_1 \in \mathcal{P} : s_1 \subseteq t \text{ and } \exists s_2 \in \mathcal{P} : t \subseteq s_2$$
$$\iff t \in \mathcal{P} \text{ (if } \mathcal{P} \in \mathbb{C})$$

Can we simply extend $\mathbf{PL}_{\vee}(\text{NE})$ to get $\phi^D_{\mathcal{P}}$? No. It can be shown that if a logic L can define $||\phi \vee \psi||$ for all convex ϕ, ψ (notation: $\mathbb{C} \vee \mathbb{C} \subseteq ||L||$) , then $||L|| \subsetneq \mathbb{C}$ (the logic cannot be convex!)

For instance, let $\mathcal{P}_1 := \{\{v_1\}, \{v_2, v_3\}\}$ and $\mathcal{P}_2 := \{\{v_1\}\}$. Then $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{C}$, so $\mathcal{P}_1 = ||\phi_1||$ and $\mathcal{P}_2 = ||\phi_2||$ for $\phi_1, \phi_2 \in L$. We have $||\phi_1 \vee \phi_2|| = \{\{v_1\}, \{v_1, v_2, v_3\}\} \notin \mathbb{C}$, so if $\mathbb{C} \vee \mathbb{C} \subseteq ||L||$, then $||L|| \subsetneq \mathbb{C}$.

We had $\vee$ in $\mathbf{PL}_{\vee}(\text{NE})$, but $\mathbf{PL}_{\vee}(\text{NE})$ can only define $\phi \vee \psi$ for all convex *and union-closed* ϕ, ψ; this does not violate convexity. $\mathbb{CU} \vee \mathbb{CU} \subseteq ||L||$ need not imply $\mathbb{C} \vee \mathbb{C} \subseteq ||L||$.

We must either (1) modify $\vee$ to force convexity, or (2) replace $\vee$ with something else (that still allows us to capture all of classical propositional logic). Recall that propositional dependence logic and propositional inquisitive logic are complete for $\mathbb{D}$ and hence can express $\phi^D_{\mathcal{P}}$. We employ strategy (1) to produce a convex extension of propositional dependence logic, and (2) to produce a convex logic similar to propositional inquisitive logic.

Convex Propositional Dependence Logic

Syntax of propositional dependence logic $\mathbf{PL}_\vee(=(\cdot))$:

$$\phi ::= p \mid \bot \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \ = (p_1, \ldots, p_n, q)$$

where $\alpha \in \mathbf{PL}_\vee$. $\|\mathbf{PL}_\vee(=(\cdot))\| = \mathbb{D}$, so $\|\phi^D_\mathcal{P}\| \in \|PL_\vee(=(\cdot))\|$.
We modify $\vee$ to force downward closure, and hence convexity. We also replace NE with the epistemic might operator $\blacklozenge$ to still be able to express $\phi^U_\mathcal{P}$.
Syntax of classical propositional logic (with $\curlyvee$) $\mathbf{PL}_\curlyvee$:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \curlyvee \alpha$$

Syntax of convex propositional dependence logic $\mathbf{PL}_\curlyvee(=(\cdot), \blacklozenge)$:

$$\phi ::= p \mid \bot \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \curlyvee \phi \mid \ = (p_1, \ldots, p_n, q) \mid \blacklozenge\phi$$

where $\alpha \in \mathbf{PL}_\curlyvee$.

$$t \vDash \phi \curlyvee \psi \iff \exists s \supseteq t : s = s_1 \cup s_2 \;\&\; s_1 \vDash \phi \;\&\; s_2 \vDash \psi$$

$$t \vDash \blacklozenge\phi \iff \exists s \subseteq t : s \neq \varnothing \;\&\; s \vDash \phi$$

For downward-closed $\phi, \psi : \phi \vee \psi \equiv \phi \curlyvee \psi$, so $\|\phi^D_\mathcal{P}\| \in \|PL_\curlyvee(=(\cdot), \blacklozenge)\|$. We can define χ_t using $\curlyvee$, and define $\phi^U_\mathcal{P}$ for $\mathcal{P} = \{t_1, \ldots, t_n\}$ by: $\phi^U_\mathcal{P} := \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \curlyvee \ldots \curlyvee \chi_{v_n})$.

A Convex Logic Similar to Propositional Inquisitive Logic

Syntax of classical propositional logic (with $\to$) $\mathbf{PL}_{\to}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

Syntax of propositional inquisitive logic $\mathbf{PL}_{\to}(\,\mathbb{V}\,)$:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \,\mathbb{V}\, \phi$$

$$t \models \phi \to \psi \iff \forall s \subseteq t : s \models \phi \text{ implies } s \models \psi$$
$$t \models \phi \,\mathbb{V}\, \psi \iff t \models \phi \text{ or } t \models \psi$$

Like $\mathbf{PL}_{\vee}$, $\mathbf{PL}_{\to}$ is flat, and corresponds to standard classical propositional logic. We define $\neg_i \phi := \phi \to \bot$. $\phi \vee_i \psi := \neg_i(\neg_i \phi \wedge \neg_i \psi)$. Using these, we can construct χ_t as before. $\|\mathbf{PL}_{\to}(\,\mathbb{V}\,)\| = \mathbb{D}$, and $\phi^D_{\mathcal{P}}$ is definable as

$$\phi^D_{\mathcal{P}} := \bigvee\!\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t$$

We again add the epistemic modality $\blacklozenge$ to capture $\phi^U_{\mathcal{P}}$:

$$\phi^U_{\mathcal{P}} := \bigwedge_{v_1 \in t_1, \dots, v_n \in t_n} \blacklozenge(\chi_{v_1} \vee_i \dots \vee_i \chi_{v_n}) \qquad (\mathcal{P} = \{t_1, \dots, t_n\})$$

Problem: with $\blacklozenge$ and $\mathbb{V}$, the logic is no longer convex. If $\mathbb{C} \mathbb{V} \mathbb{C} \subseteq ||L||$, then $||L|| \subsetneq \mathbb{C}$. E.g., $\blacklozenge p \mathbb{V} q$ is not convex.

Solution: We can have $\mathbb{F} \mathbb{V} \mathbb{F} \subseteq ||L||$ (where $\mathbb{F}$ is the class of flat properties) and hence $||\phi^D_{\mathcal{P}}|| = ||\bigvee\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t|| \in ||L||$ without having $\mathbb{V}$ in the syntax. In fact, $\mathbb{V}$ is already uniformly definable for flat ϕ, ψ using $\rightarrow$ and $\blacklozenge$.

Syntax of **PL$_{\rightarrow}(\blacklozenge)$**:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \rightarrow \phi \mid \blacklozenge \phi$$

For any $\{\alpha_k \mid k \in K\} \subseteq \mathbf{PL}_{\rightarrow}$,

$$\overline{\bigvee\!\!\!\bigvee_{k \in K}} \alpha_k := \bigwedge_{k \in K} \big((\bigwedge_{j \in K \setminus \{k\}} \blacklozenge \neg_i \alpha_j) \rightarrow \alpha_k \big).$$

Then $\overline{\bigvee\!\!\!\bigvee}_{k \in K} \alpha_k \equiv \bigvee\!\!\!\bigvee_{k \in K} \alpha_k$. We can define $\phi^U_{\mathcal{P}}$ as before, and $\phi^D_{\mathcal{P}}$ as:

$$\phi^D_{\mathcal{P}} := \overline{\bigvee\!\!\!\bigvee_{t \in \mathcal{P}}} \chi_t$$

Conclusion

- Importance of convexity.
- Notion of propositionhood in team logics.
- Results: $\mathbf{PL}_{\vee}(\mathrm{NE})$ is expressively complete for convex and union-closed properties. A modal analogue of the result shows that Aloni's BSML is expressively complete for modal convex and union-closed properties.
- Results: Two logics expressively complete for all convex properties. One is similar to propositional dependence logic, the other to propositional inquisitive logic.

Thank you!