# Deep Inference Sequent Calculi for Propositional Logics with Team Semantics

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PhDs in Logic XIV

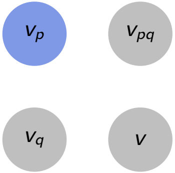
# Team semantics

In team semantics, formulas are interpreted with respect sets of valuations—teams—rather than single valuations.

**single-valuation semantics**

$v \models \phi$

$v \in 2^{Prop}$

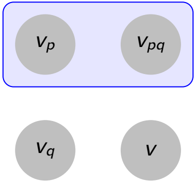



$v_p \models p$

**team semantics**

$s \models \phi$

$s \subseteq 2^{Prop}$




$\{v_p, v_{pq}\} \models p$

# Team semantics

In team semantics, formulas are interpreted with respect sets of valuations—teams—rather than single valuations. Teams provide for ways to express meanings not readily expressible in single-valuation semantics.

single-valuation semantics

$v \models \phi$

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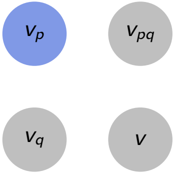


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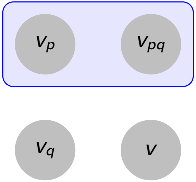


$\{v_p, v_{pq}\} \models p$

dependence logic example:

| | $p$ | $q$ | $r$ |
|---|---|---|---|
| $v_1$ | 0 | 1 | 1 |
| $v_2$ | 0 | 1 | 0 |
| $v_3$ | 1 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 |

$s \models D(p,q)$ $s \not\models D(p,r)$

the value of p determines the value of q but does not determine the value of r

# $PL(\mathbb{V})$

Syntax of classical propositional logic *CPL*:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

Syntax of propositional logic with the global/inquisitive disjunction $\mathbb{V}$ $PL(\mathbb{V})$

$$\phi ::= p \mid \bot \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbb{V} \phi \quad \text{where } \alpha \in CPL$$

$PL(\mathbb{V})$ is expressively equivalent to propositional dependence logic and propositional inquisitive logic.

# Semantics

| | | |
|---|---|---|
| $s \vDash p$ | $\iff$ | $\forall v \in s : v(p) = 1$ |
| $s \vDash \bot$ | $\iff$ | $s = \varnothing$ |
| $s \vDash \neg\alpha$ | $\iff$ | $\forall v \in s : \{v\} \nvDash \alpha$ |
| $s \vDash \phi \vee \psi$ | $\iff$ | $\exists t, t' : t \cup t' = s$ & $t \vDash \phi$ & $t' \vDash \psi$ |
| $s \vDash \phi \wedge \psi$ | $\iff$ | $s \vDash \phi$ and $s \vDash \psi$ |
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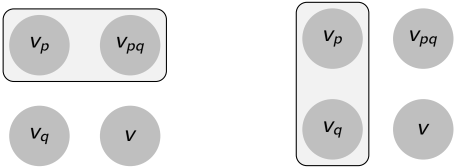


(a) $s \vDash p$ $s \vDash \neg r$

(b) $s \nvDash p$

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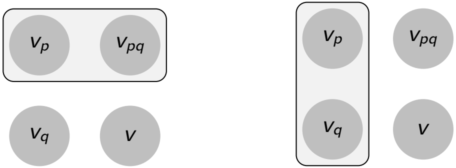
$s \vDash \phi \vee \psi \iff \exists t, t' : t \cup t' = s \ \& \ t \vDash \phi \ \& \ t' \vDash \psi$

$s \vDash \phi \wedge \psi \iff s \vDash \phi$ and $s \vDash \psi$

$s \vDash \phi \mathbb{V} \psi \iff s \vDash \phi$ or $s \vDash \psi$

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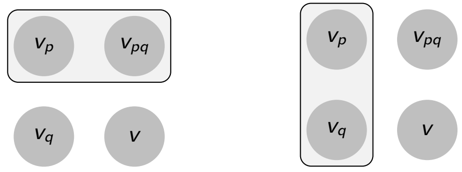

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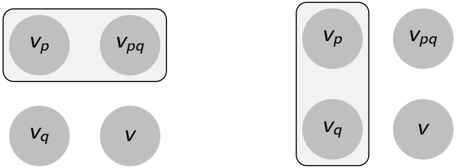


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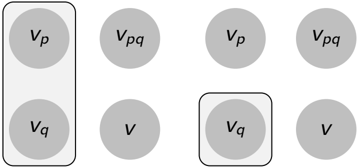


(c) $s \vDash p \vee q$ (d) $s \vDash p \vee q$

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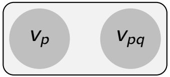

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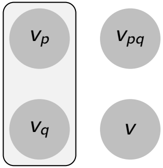
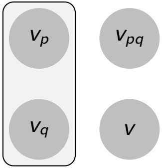


(b) $s \nvDash p$



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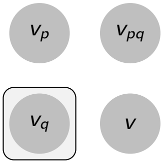

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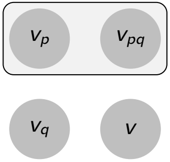


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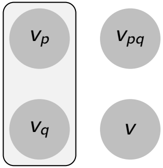
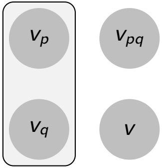



(b) $s \nvDash p$

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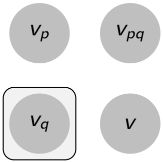


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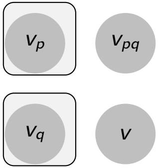


(e) $\{v_p\} \vDash p \mathbb{V} \neg p$
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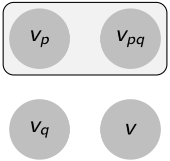


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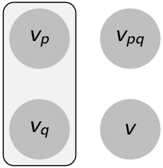


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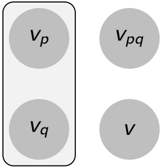


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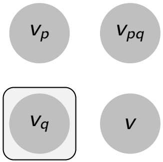


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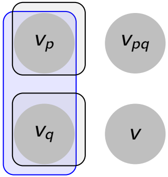


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# Closure properties

$\phi$ is *downward closed*: $[s \models \phi \text{ and } t \subseteq s] \implies t \models \phi$

$\phi$ is *union closed*: $[s \models \phi \text{ for all } s \in S \neq \varnothing] \implies \bigcup S \models \phi$

$\phi$ has the *empty team property* : $\varnothing \models \phi$

$\phi$ is *flat* : $s \models \phi \iff \{v\} \models \phi \text{ for all } v \in s$

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flat $\iff$ downward closed & union closed & empty team property

*CPL*-formulas are flat and their team semantics coincide with their standard semantics on singletons:

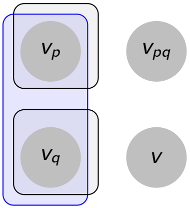
$$\text{for } \alpha \in CPL: \quad s \models \alpha \iff \forall v \in s : \{v\} \models \alpha \iff \forall v \in s : v \models \alpha$$

Therefore $PL(\mathbb{V})$ is a conservative extension of classical propositional logic:

$$\text{for } \Xi \cup \{\alpha\} \subseteq CPL: \quad \Xi \models \alpha \text{ (in team semantics)} \iff \Xi \models \alpha \text{ (in standard semantics)}$$

# The global/inquisitive disjunction $\mathbb{V}$

All formulas are downward closed and have the empty team property, but formulas with $\mathbb{V}$ might not be union closed.

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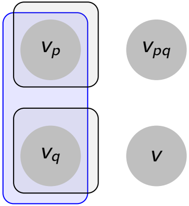


$$
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\{v_p, v_{\neg p}\} & \models & (p \mathbb{V} \neg p) \vee (p \mathbb{V} \neg p)
\end{array}
$$

$PL(\mathbb{V})$ is not closed under uniform substitution.
E.g., $p \vee p \models p$ but $(p \mathbb{V} \neg p) \vee (p \mathbb{V} \neg p) \not\models p \mathbb{V} \neg p$.

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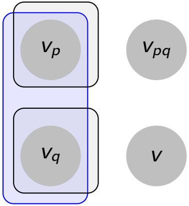


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$\wedge$, $\vee$, and $\mathbb{V}$ distribute over $\mathbb{V}$:

$$\begin{array}{lll} \phi \wedge (\psi \mathbb{V} \chi) & \equiv & (\phi \wedge \psi) \mathbb{V} (\phi \wedge \chi) \\ \phi \vee (\psi \mathbb{V} \chi) & \equiv & (\phi \vee \psi) \mathbb{V} (\phi \vee \chi) \\ \phi \mathbb{V} (\psi \mathbb{V} \chi) & \equiv & (\phi \mathbb{V} \psi) \mathbb{V} (\phi \mathbb{V} \chi) \end{array}$$

Therefore, each $\phi \in PL(\mathbb{V})$ is equivalent to a $\mathbb{V}$-disjunction of classical formulas called the resolutions of $\phi$: $\phi \equiv \mathbb{V} R(\phi)$ ($R(\phi) \subseteq CPL$).

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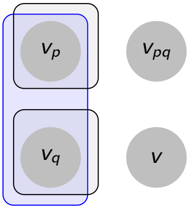


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**Split property**

For $\Xi \subseteq CPL$:

$$\Xi \models \phi_1 \mathbb{V} \phi_2 \text{ iff } \Xi \models \phi_1 \text{ or } \Xi \models \phi_2.$$

# Natural deduction system

$\alpha$ must be classical.

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\wedge I \qquad \frac{\phi \wedge \psi}{\phi}\wedge E \qquad \frac{\phi \wedge \psi}{\psi}\wedge E$$

$$\frac{\begin{matrix}[\alpha]\\ \vdots \\ \bot\end{matrix}}{\neg\alpha}\neg I \qquad \frac{\alpha \quad \neg\alpha}{\phi}\neg E \qquad \frac{\begin{matrix}[\neg\alpha]\\ \vdots \\ \bot\end{matrix}}{\alpha}\mathrm{RAA} \qquad \frac{\bot}{\phi}\mathrm{EF}$$

$$\frac{\phi}{\phi \mathbb{V} \psi}\mathbb{V}I \qquad \frac{\phi}{\psi \mathbb{V} \phi}\mathbb{V}I \qquad \frac{\phi \mathbb{V} \psi \quad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\mathbb{V}E$$

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$$\frac{\phi}{\phi \mathbb{V} \psi}\mathbb{V}\text{I} \qquad \frac{\phi}{\psi \mathbb{V} \phi}\mathbb{V}\text{I} \qquad \frac{\phi \mathbb{V} \psi \qquad \begin{array}{c}[\phi]\\ \vdots\\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\chi}\mathbb{V}\text{E}$$

$$\frac{\phi}{\phi \vee \psi}\vee\text{I} \qquad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \alpha\end{array}}{\alpha}\vee\text{E}$$

$$\frac{\phi \vee \psi}{\psi \vee \phi}\vee\text{Com} \qquad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots\\ \chi\end{array}}{\chi \vee \psi}\vee\text{Mon}$$

$$\frac{\phi \vee (\psi \mathbb{V} \chi)}{(\phi \vee \psi) \mathbb{V} (\phi \vee \chi)}\text{Dstr}\vee\mathbb{V}$$

# A sequent calculus for *CPL*

*Axioms*

$$\Gamma, p \Rightarrow p, \Delta \quad At \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

*Logical rules*

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta}\, L\neg \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}\, R\neg$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\, R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\, R\vee$$

*Cut*

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Pi, \phi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\, Cut$$

G3-style sequent calculus for *CPL*.

A naive translation of the ND-system:

$$\Gamma, p \Rightarrow p, \Delta \quad At \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\ L\neg \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \phi, \Xi \qquad \Gamma \Rightarrow \psi, \Xi}{\Gamma \Rightarrow \phi \wedge \psi, \Xi, \Delta}\ R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Xi \qquad \Gamma, \psi \Rightarrow \Xi}{\Gamma, \phi \vee \psi \Rightarrow \Xi, \Delta}\ L\vee \qquad\qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\ R\vee$$

$$\frac{\Gamma, \phi_1 \Rightarrow \Delta \qquad \Gamma, \phi_2 \Rightarrow \Delta}{\Gamma, \phi_1 \mathbb{V} \phi_2 \Rightarrow \Delta}\ L\mathbb{V} \qquad\qquad \frac{\Gamma \Rightarrow \phi_i, \Delta}{\Gamma \Rightarrow \phi_1 \mathbb{V} \phi_2, \Delta}\ R\mathbb{V}$$

$$\frac{\Gamma, \phi \vee \psi_1 \Rightarrow \Delta \qquad \Gamma, \phi \vee \psi_2 \Rightarrow \Delta}{\Gamma, \phi \vee (\psi_1 \mathbb{V} \psi_2) \Rightarrow \Delta}\ LDstr \qquad\qquad \frac{\Gamma \Rightarrow \phi \vee (\psi_1 \mathbb{V} \psi_2), \Delta}{\Gamma \Rightarrow (\phi \vee \psi_1) \mathbb{V} (\phi \vee \psi_2), \Delta}\ RDstr$$

$\alpha, \Xi$ must be classical.

Problem 1: the distributivity rules are not strong enough—how would one give a cutfree proof of the following sequent in this system?

$$(((p \wedge x) \mathbb{V} (q \wedge x)) \vee (y \wedge x)) \vee (r \wedge x) \Rightarrow (((p \vee y) \vee r) \wedge x) \mathbb{V} (((q \vee y) \vee r) \wedge x)$$

Problem 2: How does cut elimination work with the restricted rules?
In a classical cut elimination proof, the cut below

$$\dfrac{\dfrac{\begin{matrix}D_1' \\ \Gamma, \eta \Rightarrow \phi, \Xi\end{matrix} \qquad \begin{matrix}D_1' \\ \Gamma, \xi \Rightarrow \phi, \Xi\end{matrix}}{\Gamma, \eta \vee \xi \Rightarrow \phi, \Xi}\, L\vee \qquad \begin{matrix}D_2' \\ \Pi, \phi \Rightarrow \Lambda\end{matrix}}{\Pi, \Gamma, \eta \vee \xi \Rightarrow \Xi, \Lambda}\, \text{Cut}$$

can be commuted upwards:

$$\dfrac{\dfrac{\begin{matrix}D_1' \\ \Gamma, \eta \Rightarrow \phi, \Xi\end{matrix} \qquad \begin{matrix}D_2' \\ \Pi, \phi \Rightarrow \Sigma\end{matrix}}{\Pi, \Gamma, \eta \Rightarrow \Xi, \Lambda}\, \text{Cut} \qquad \dfrac{\begin{matrix}D_1' \\ \Gamma, \xi \Rightarrow \phi, \Xi\end{matrix} \qquad \begin{matrix}D_2' \\ \Pi, \phi \Rightarrow \Lambda\end{matrix}}{\Pi, \Gamma, \xi \Rightarrow \Xi, \Lambda}\, \text{Cut}}{\Pi, \Gamma, \eta \vee \xi \Rightarrow \Xi, \Lambda}\, L\vee$$

If there are restrictions on the rules, this cannot be done freely:

$$\dfrac{\dfrac{\begin{array}{c}D_1'\\ \Gamma,\eta\Rightarrow\phi,\Xi\end{array} \qquad \begin{array}{c}D_1'\\ \Gamma,\xi\Rightarrow\phi,\Xi\end{array}}{\Gamma,\eta\vee\xi\Rightarrow\phi,\Xi,\Delta}\,L\vee \qquad \begin{array}{c}D_2'\\ \Pi,\phi\Rightarrow\Sigma\end{array}}{\Pi,\Gamma,\eta\vee\xi\Rightarrow\Xi,\Delta,\Sigma}\,\text{Cut}$$

would be transformed into

$$\dfrac{\dfrac{\begin{array}{c}D_1'\\ \Gamma,\eta\Rightarrow\phi,\Xi\end{array} \qquad \begin{array}{c}D_2'\\ \Pi,\phi\Rightarrow\Sigma\end{array}}{\Pi,\Gamma,\eta\Rightarrow\Xi,\Sigma}\,\text{Cut} \qquad \dfrac{\begin{array}{c}D_1'\\ \Gamma,\xi\Rightarrow\phi,\Xi\end{array} \qquad \begin{array}{c}D_2'\\ \Pi,\phi\Rightarrow\Sigma\end{array}}{\Pi,\Gamma,\xi\Rightarrow\Xi,\Sigma}\,\text{Cut}}{\Pi,\Gamma,\eta\vee\xi\Rightarrow\Xi,\Sigma,\Delta}\,\#L\vee$$

which contains an illegitimate application of $L\vee$.

# A deep inference system

*Axioms*

$$\Gamma, p \Rightarrow p, \Delta \quad At \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

*Logical rules*

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\, L\neg \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\, R\neg$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \phi, \Xi \qquad \Gamma \Rightarrow \psi, \Xi}{\Gamma \Rightarrow \phi \wedge \psi, \Xi, \Delta}\, R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Xi \qquad \Gamma, \psi \Rightarrow \Xi}{\Gamma, \phi \vee \psi \Rightarrow \Xi, \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\, R\vee$$

$$\frac{\Gamma, \chi[\phi_1/\eta] \Rightarrow \Delta \qquad \Gamma, \chi[\phi_2/\eta] \Rightarrow \Delta}{\Gamma, \chi[\phi_1 \bigvee \phi_2/\eta] \Rightarrow \Delta}\, L\bigvee \qquad \frac{\Gamma \Rightarrow \chi[\phi_i/\eta], \Delta}{\Gamma \Rightarrow \chi[\phi_1 \bigvee \phi_2/\eta], \Delta}\, R\bigvee$$

The intended interpretation of $\Gamma \Rightarrow \Delta$ is $\bigwedge \Gamma \models \bigvee \Delta$ (not $\bigwedge \Gamma \models \bigvee\!\!\!\bigvee \Delta$).

$\alpha, \Xi$ must be classical.

$\phi[\psi]$: a *specific* occurrence of the subformula $\psi$ within $\phi$.

$\phi[\chi/\psi]$: the result of replacing $\phi[\psi]$ in $\phi$ with $\chi$.

$\eta$ must not occur within the scope of a negation.

Alternative multiplicative linear logic-style rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma_1 \Rightarrow \phi, \Delta_1 \qquad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \phi \wedge \psi, \Delta_1, \Delta_2}\ R\wedge$$

$$\frac{\Gamma_1, \phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \vee \psi \Rightarrow \Delta_1, \Delta_2}\ L\vee \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\ R\vee$$

*Structural rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\ RW$$

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \alpha, \Delta}\ RC$$

# Weak subformula property

**Definition ( $\mathbb{V}$-subformulas)**

The set $SF_{\mathbb{V}}(\phi)$ of $\mathbb{V}$*-subformulas* of $\phi$ is defined recursively by:

$$SF_{\mathbb{V}}(p) = \{p\}$$
$$SF_{\mathbb{V}}(\neg\alpha) = \{\neg\alpha\} \cup SF_{\mathbb{V}}(\alpha)$$
$$SF_{\mathbb{V}}(\phi \wedge \psi) = \{\phi \wedge \psi\} \cup SF_{\mathbb{V}}(\phi) \cup SF_{\mathbb{V}}(\psi)$$
$$SF_{\mathbb{V}}(\phi \vee \psi) = \{\phi \vee \psi\} \cup SF_{\mathbb{V}}(\phi) \cup SF_{\mathbb{V}}(\psi)$$
$$SF_{\mathbb{V}}(\chi[\phi_1 \mathbb{V} \phi_2/\eta]) = \{\chi[\phi_1 \mathbb{V} \phi_2/\eta]\} \cup SF_{\mathbb{V}}(\chi[\phi_1/\eta]) \cup SF_{\mathbb{V}}(\chi[\phi_2/\eta])$$

**Weak subformula property**

Any formulas appearing in a cutfree proof of $\Gamma \Rightarrow \Delta$ are $SF_{\mathbb{V}}$-subformulas of formulas in $\Gamma, \Delta$.

# Depth-preserving weakening, contraction and inversion; Interpolation

$\vdash_n S$: $S$ has a derivation of depth at most $n$.

**Weakening and contraction lemma**

If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma, \phi \Rightarrow \Delta$

If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma \Rightarrow \phi, \Delta$

If $\vdash_n \Gamma, \phi, \phi \Rightarrow \Delta$ then $\vdash_n \Gamma, \phi \Rightarrow \Delta$

If $\vdash_n \Gamma \Rightarrow \alpha, \alpha, \Delta$ then $\vdash_n \Gamma \Rightarrow \alpha, \Delta$

Right contraction is not sound with respect to all formulas since, e.g., $(p \mathbin{\mathbb{V}} \neg p) \vee (p \mathbin{\mathbb{V}} \neg p) \not\models p \mathbin{\mathbb{V}} \neg p$.

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**Inversion lemma**

All rules except $R\mathbb{V}$ are depth-preserving invertible.

E.g., $(R\wedge)$ $\vdash_n \Gamma \Rightarrow \phi \wedge \psi, \Delta$ implies $\vdash_n \Gamma \Rightarrow \phi, \Delta$ and $\vdash_n \Gamma \Rightarrow \psi, \Delta$.

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Let $\Gamma_1; \Gamma_2$ be a partition of $\Gamma$ and $\Delta_1; \Delta_2$ be a partition of $\Delta$. $I$ is a sequent interpolant of $\Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2$ if $I$ is in the language $(\Gamma_1 \cup \Delta_1) \cap (\Gamma_2 \cup \Delta_2)$; if $\vdash \Gamma_1 \Rightarrow I, \Delta_1$; and if $\vdash \Gamma_2, I \Rightarrow \Delta_2$.

**Constructive proof of interpolation**

If $\vdash \Gamma \Rightarrow \Delta$, then for each pair of partitions $\Gamma_1; \Gamma_2, \Delta_1; \Delta_2$ for $\Gamma \Rightarrow \Delta$, there is a sequent interpolant $I$ of $\Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2$, and if $\Delta_2$ is classical, then $I$ is classical.

# Countermodel search

One simple way of proving cutfree completeness of the system (similar to a proof for the classical G3 system) relies on the (semantic) invertibility of the rules and on the split property.

**Countermodel search/completeness**

There is a procedure that constructs a countermodel to $\Gamma \Rightarrow \Delta$ from countermodels to sequents only involving atomic formulas if there is such a countermodel. If there is no such countermodel, the procedure yields a cutfree proof of $\Gamma \Rightarrow \Delta$.

An example, with countermodels written above the sequent arrows:

$$
\dfrac{
\dfrac{p \Rightarrow p \qquad \dfrac{p, p \overset{\{v_p\}}{\Rightarrow}}{p \overset{\{v_p\}}{\Rightarrow} \neg p}\,R\neg}{p \Rightarrow p \Vvee \neg p}\,R\Vvee
\qquad
\dfrac{
\dfrac{p \Rightarrow p \qquad \dfrac{\overset{\{v_{\overline{p}}\}}{\Rightarrow} p, p}{\neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p}\,L\neg}{p \vee \neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p}\,L\vee
\qquad
\dfrac{\dfrac{p, p \overset{\{v_p\}}{\Rightarrow}}{p \overset{\{v_p\}}{\Rightarrow} \neg p}\,R\neg \qquad \dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,L\neg}{\neg p \Rightarrow \neg p}\,R\neg}{p \vee \neg p \overset{\{v_p\}}{\Rightarrow} \neg p}\,L\vee
}{p \vee \neg p \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \Vvee \neg p}\,R\Vvee
}{p \Vvee (p \vee \neg p) \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \Vvee \neg p}\,L\Vvee
$$

$$
\dfrac{\Xi \Rightarrow \chi[\phi_1], \Delta \qquad \Xi \Rightarrow \chi[\phi_2], \Delta}{\Xi \Rightarrow \chi[\phi_1 \Vvee \phi_2], \Delta}\,R\Vvee
$$

denotes that $\Xi \Rightarrow \chi[\phi_1 \Vvee \phi_2], \Delta$ holds iff either $\Xi \Rightarrow \chi[\phi_1], \Delta$ or $\Xi \Rightarrow \chi[\phi_2], \Delta$ holds.

# Normal form for cutfree derivations

The notation $\begin{array}{c} X \\ \| A \\ Y \end{array}$ means that the set of sequents/single sequent $Y$ is derivable from the set of sequents/single sequent $X$ using rules $A$.

## Normal form for cutfree derivations

$$\begin{array}{c} \| PL(\mathbb{V}) \\ \Gamma \Rightarrow \Delta \end{array} \text{ implies there is a } f : R(\Gamma) \to R(\Delta) \text{ s.t. } \begin{array}{c} \| CPL \\ \{\Xi \Rightarrow f(\Xi) \mid \Xi \in R(\Gamma)\} \\ \| R\mathbb{V} \\ \{\Xi \Rightarrow \Delta \mid \Xi \in R(\Gamma)\} \\ \| L\mathbb{V} \\ \Gamma \Rightarrow \Delta \end{array}$$

# Example derivation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
p \Rightarrow p, q \wedge \neg r
\qquad
\dfrac{
\dfrac{
q, \neg r \Rightarrow p, q
\qquad
\dfrac{
\dfrac{q, r \Rightarrow r, p}{q, \neg r, r \Rightarrow p}\, L\neg
}{q, \neg r \Rightarrow p, \neg r}\, R\neg
}{q, \neg r \Rightarrow p, q \wedge \neg r}\, R\wedge
}{q \wedge \neg r \Rightarrow p, q \wedge \neg r}\, L\wedge
}{p \vee (q \wedge \neg r) \Rightarrow p, q \wedge \neg r}\, L\vee
}{p \vee (q \wedge \neg r) \Rightarrow p \vee (q \wedge \neg r)}\, R\vee
}{p \vee (q \wedge \neg r) \Rightarrow (p \vee (q \wedge \neg r)) \mathbb{V} (p \vee (q \wedge s))}\, R\mathbb{V}
\qquad
\dfrac{
\dfrac{
\dfrac{
p \Rightarrow p, q \wedge s
\qquad
\dfrac{
\dfrac{q, s \Rightarrow p, q \qquad q, s \Rightarrow p, s}{q, s \Rightarrow p, q \wedge s}\, R\wedge
}{q \wedge s \Rightarrow p, q \wedge s}\, L\wedge
}{p \vee (q \wedge s) \Rightarrow p, q \wedge s}\, L\vee
}{p \vee (q \wedge s) \Rightarrow p \vee (q \wedge s)}\, R\vee
}{p \vee (q \wedge s) \Rightarrow (p \vee (q \wedge \neg r)) \mathbb{V} (p \vee (q \wedge s))}\, R\mathbb{V}
}{p \vee (q \wedge (\neg r \mathbb{V} s)) \Rightarrow (p \vee (q \wedge \neg r)) \mathbb{V} (p \vee (q \wedge s))}\, L\mathbb{V}
$$

# Cut elimination

Given a cut

$$\dfrac{\overset{D_1}{\Gamma \Rightarrow \phi, \Delta} \qquad \overset{D_2}{\Pi, \phi \Rightarrow \Sigma}}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ \text{Cut}$$

where $D_1$ and $D_2$ are cutfree, apply the normal form theorem to $D_1$ and $D_2$ to obtain $f : R(\Gamma) \to R(\Delta, \phi), g : R(\Pi, \phi) \to R(\Sigma)$ such that

$$\begin{array}{cc}
\| \mathit{CPL} & \| \mathit{CPL} \\
\{\Xi \Rightarrow f(\Xi) \mid \Xi \in R(\Gamma)\} & \{\Lambda \Rightarrow g(\Lambda) \mid \Lambda \in R(\Pi, \phi)\} \\
\| R \mathbb{V} & \| R \mathbb{V} \\
\{\Xi \Rightarrow \Delta, \phi \mid \Xi \in R(\Gamma)\} & \{\Lambda \Rightarrow \Sigma \mid \Lambda \in R(\Pi, \phi)\} \\
\| L \mathbb{V} & \| L \mathbb{V} \\
\Gamma \Rightarrow \Delta, \phi & \Pi, \phi \Rightarrow \Sigma
\end{array}$$

For each $\Xi \in R(\Gamma)$ and each $\Lambda \in R(\Pi)$ there is $\alpha_{\Xi,\Lambda} \in R(\phi)$ s.t. $\alpha_{\Xi,\Lambda} \in f(\Xi)$ and $\{\Lambda, \alpha_{\Xi,\Lambda}\} \in R(\Pi, \phi)$ so:

$$\dfrac{\overset{D_a}{\Xi \Rightarrow f(\Xi)', \alpha_{\Xi,\Lambda}} \qquad \overset{D_b}{\Lambda, \alpha_{\Xi,\Lambda} \Rightarrow g(\Lambda, \alpha_{\Xi,\Lambda})}}{\Xi, \Lambda \Rightarrow f(\Xi)', g(\Lambda, \alpha_{\Xi,\Lambda})}\ \text{Cut}$$

where $D_a$ and $D_b$ are classical (and $f(\Xi) = f(\Xi)', \alpha_{\Xi,\Lambda}$). By classical cut elimination, there is then also a classical cutfree derivation of $\Xi, \Lambda \Rightarrow f(\Xi)', g(\Lambda, \alpha)$. One can then show:

$$\begin{array}{c}
\| \mathit{CPL} \\
\{\Xi, \Lambda \Rightarrow f(\Xi)', g(\Lambda, \alpha_{\Xi,\Lambda}) \mid \Xi \in R(\Gamma), \Lambda \in R(\Pi)\} \\
\| R \mathbb{V} \\
\{\Xi, \Lambda \Rightarrow \Delta, \Sigma \mid \Xi \in R(\Gamma), \Lambda \in R(\Pi)\} \\
\| L \mathbb{V} \\
\Gamma, \Pi \Rightarrow \Delta, \Sigma
\end{array}$$