Convexity in propositional team semantics

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Helsinki Logic Seminar

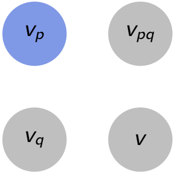
Team semantics

In team semantics, formulas are interpreted with respect sets of valuations—teams—rather than single valuations.

single-valuation semantics

$v \models \phi$

$v \in 2^{Prop}$

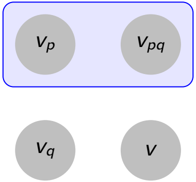


$v_p \models p$

team semantics

$s \models \phi$

$s \subseteq 2^{Prop}$



$\{v_p, v_{pq}\} \models p$

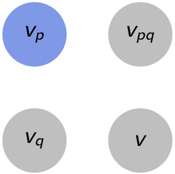
Team semantics

In team semantics, formulas are interpreted with respect sets of valuations—teams—rather than single valuations. Teams provide for ways to express meanings not readily expressible in single-valuation semantics.

single-valuation semantics

$v \vDash \phi$

$v \in 2^{Prop}$

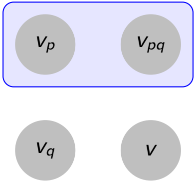


$v_p \vDash p$

team semantics

$s \vDash \phi$

$s \subseteq 2^{Prop}$



$\{v_p, v_{pq}\} \vDash p$

dependence logic example:

	p	q	r
v_1	0	1	1
v_2	0	1	0
v_3	1	0	0
v_3	1	0	0

$s \vDash =\!(p, q)$ $s \nvDash =\!(p, r)$

the value of p determines the value of q but does not determine the value of r

Syntax

Syntax of classical propositional logic **PL**

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

where $p \in Prop$ (some fixed set of propositional variables).

We consider extensions of **PL** by various non-classical connectives such as the non-emptiness atom NE and the global disjunction $\mathbin{\mathbb{V}}$. In these extension, negations are restricted to classical formulas. E.g., syntax of **PL**$(\mathrm{NE}, \mathbin{\mathbb{V}})$:

$$\phi ::= p \mid \bot \mid \mathrm{NE} \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbin{\mathbb{V}} \phi$$

where $p \in Prop, \alpha \in$ **PL**.

Semantics

$$s \models p \iff \forall v \in s : v(p) = 1$$

$$s \models \bot \iff s = \varnothing$$

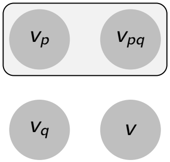
$$s \models \neg\alpha \iff \forall v \in s : \{v\} \not\models \alpha$$

$$s \models \phi \wedge \psi \iff s \models \phi \text{ and } s \models \psi$$

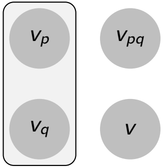
$$s \models \phi \vee \psi \iff \exists t, t' : t \cup t' = s \ \& \ t \models \phi \ \& \ t' \models \psi$$

$$s \models \mathrm{NE} \iff s \neq \varnothing$$

$$s \models \phi \mathbin{\mathbb{V}} \psi \iff s \models \phi \text{ or } s \models \psi$$



(a) $s \models p$ $s \models \neg r$



(b) $s \not\models p$

Semantics

$s \models p \iff \forall v \in s : v(p) = 1$

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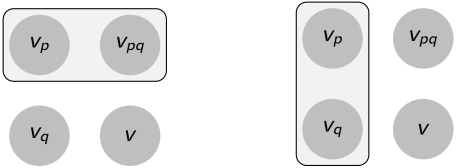
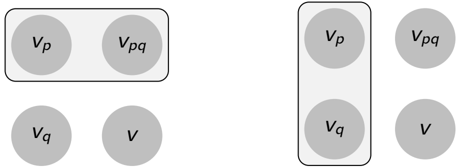


(a) $s \models p \quad s \models \neg r$

(b) $s \not\models p$

Semantics

$s \vDash p$	$\iff$	$\forall v \in s : v(p) = 1$
$s \vDash \bot$	$\iff$	$s = \varnothing$
$s \vDash \neg\alpha$	$\iff$	$\forall v \in s : \{v\} \not\vDash \alpha$
$s \vDash \phi \wedge \psi$	$\iff$	$s \vDash \phi$ and $s \vDash \psi$
$s \vDash \phi \vee \psi$	$\iff$	$\exists t, t' : t \cup t' = s$ & $t \vDash \phi$ & $t' \vDash \psi$
$s \vDash \mathrm{NE}$	$\iff$	$s \neq \varnothing$
$s \vDash \phi \mathbin{\mathbb{V}} \psi$	$\iff$	$s \vDash \phi$ or $s \vDash \psi$



(a) $s \vDash p$ $s \vDash \neg r$

(b) $s \not\vDash p$

Semantics

$s \vDash p$	$\Longleftrightarrow$	$\forall v \in s : v(p) = 1$
$s \vDash \bot$	$\Longleftrightarrow$	$s = \varnothing$
$s \vDash \neg\alpha$	$\Longleftrightarrow$	$\forall v \in s : \{v\} \nvDash \alpha$
$s \vDash \phi \wedge \psi$	$\Longleftrightarrow$	$s \vDash \phi$ and $s \vDash \psi$
$s \vDash \phi \vee \psi$	$\Longleftrightarrow$	$\exists t, t' : t \cup t' = s$ & $t \vDash \phi$ & $t' \vDash \psi$
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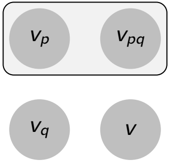

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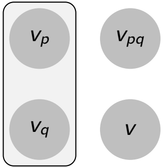

(b) $s \nvDash p$

Semantics

$$s \models p \iff \forall v \in s : v(p) = 1$$

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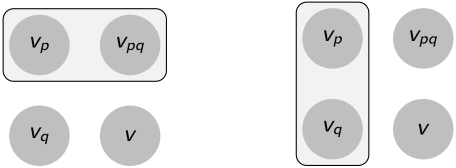
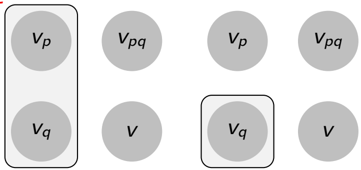


(a) $s \models p \quad s \models \neg r$ (b) $s \not\models p$



(c) $s \models p \vee q$ (d) $s \models p \vee q$

Semantics

$s \models p$	$\iff$	$\forall v \in s : v(p) = 1$
$s \models \bot$	$\iff$	$s = \varnothing$
$s \models \neg\alpha$	$\iff$	$\forall v \in s : \{v\} \not\models \alpha$
$s \models \phi \wedge \psi$	$\iff$	$s \models \phi$ and $s \models \psi$
$s \models \phi \vee \psi$	$\iff$	$\exists t, t' : t \cup t' = s$ & $t \models \phi$ & $t' \models \psi$
$s \models \mathrm{NE}$	$\iff$	$s \neq \varnothing$
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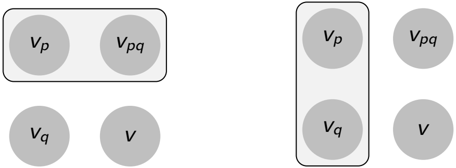

(a) $s \models p$ $s \models \neg r$

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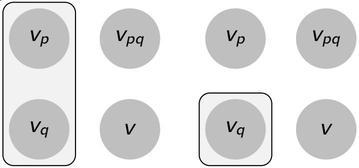

(c) $s \models p \vee q$
$s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

(d) $s \models p \vee q$
$s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

Semantics

$s \models p \iff \forall v \in s : v(p) = 1$

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$s \models \neg\alpha \iff \forall v \in s : \{v\} \not\models \alpha$

$s \models \phi \wedge \psi \iff s \models \phi$ and $s \models \psi$

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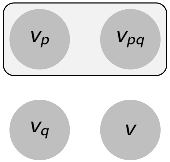

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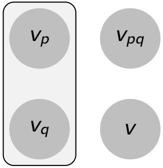
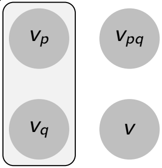


(b) $s \not\models p$



(c) $s \models p \vee q$
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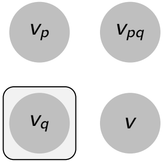
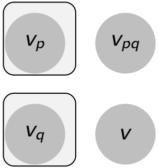


(d) $s \models p \vee q$
$s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$



(e) $\{v_p\} \models p \mathbin{\mathbb{V}} \neg p$
$\{v_q\} \models p \mathbin{\mathbb{V}} \neg p$

Semantics

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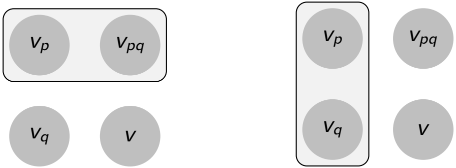

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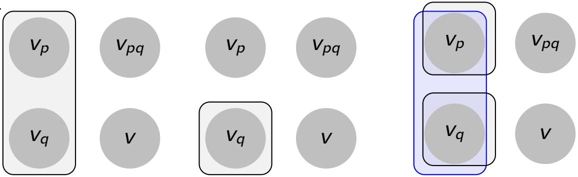

(c) $s \models p \vee q$ $s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

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(e) $\{v_p\} \models p \varovee \neg p$ $\{v_q\} \models p \varovee \neg p$

Closure properties

Definition

ϕ is *downward closed*:	$[s \models \phi \text{ and } t \subseteq s] \implies t \models \phi$
ϕ is *union closed*:	$[s \models \phi \text{ for all } s \in S \neq \varnothing] \implies \bigcup S \models \phi$
ϕ has the *empty team property*:	$\varnothing \models \phi$
ϕ is *flat*:	$s \models \phi \iff \{v\} \models \phi \text{ for all } v \in s$

flat $\iff$ downward closed & union closed & empty team property

PL-formulas are flat and their team semantics coincide with their standard semantics on singletons:

$$\text{for } \alpha \in \mathbf{PL}: \quad s \models \alpha \iff \forall v \in s : \{v\} \models \alpha \iff \forall v \in s : v \models \alpha$$

Therefore the logics we consider are conservative extensions of classical propositional logic:

$$\text{for } \Xi \cup \{\alpha\} \subseteq \mathbf{PL}: \quad \Xi \models \alpha \text{ (in team semantics)} \iff \Xi \models \alpha \text{ (in standard semantics)}$$

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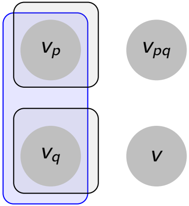
Therefore the logics we consider are conservative extensions of classical propositional logic:

$$\text{for } \Xi \cup \{\alpha\} \subseteq \mathbf{PL}: \quad \Xi \models \alpha \text{ (in team semantics)} \iff \Xi \models \alpha \text{ (in standard semantics)}$$

- Formulas with $\mathbb{V}$ might not be union closed.
- Formulas with NE might not be downward closed or have the empty team property.
- Team-based logics are commonly not closed under uniform substitution, e.g., $p \models p \vee p$ but $(p \mathbb{V} \neg p) \vee (p \mathbb{V} \neg p) \not\models p \mathbb{V} \neg p$



$$\{v_p\} \models p \mathbb{V} \neg p$$
$$\{v_q\} \models p \mathbb{V} \neg p$$
$$\{v_p, v_q\} \not\models p \mathbb{V} \neg p$$

$$\{v_p, v_q\} \models (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})$$
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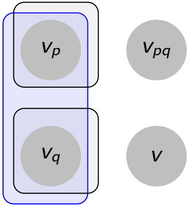

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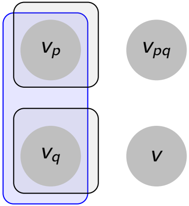

$$
\begin{array}{lcl}
\{v_p\} & \models & p \mathbb{V} \neg p \\
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 & & \\
\{v_p, v_q\} & \models & (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}) \\
\{v_p\} & \not\models & (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}) \\
 & & \\
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\end{array}
$$

Expressive Completeness

Definition

- $\phi(N)$—the propositional variables in ϕ are among $N \subseteq Prop$
- The domain of a team $s \subseteq 2^N$ is N—$dom(s) = N$. If $dom(s) = N$, we also say s is a team over N.
- A *(team) property* $\mathcal{P}$ (over N) is a class of teams (over N): $\mathcal{P} \subseteq 2^{2^N}$.
- For a formula $\phi(N)$, the property (over N) defined by ϕ is $||\phi||_N := \{s \subseteq 2^N \mid s \vDash \phi\}$.
- For a class of properties $\mathbb{P} \subseteq 2^{2^{2^{Prop}}}$ and $N \subseteq Prop$, $\mathbb{P}_N := \{\mathcal{P} \in \mathbb{P} \mid \mathcal{P} \subseteq 2^{2^N}\}$.
- A logic (or language) $\mathcal{L}$ is *expressively complete* for a class of properties $\mathbb{P}$ iff for each finite $N \subseteq Prop$:

$$||\mathcal{L}||_N := \{||\phi(N)||_N \mid \phi \in \mathcal{L}\} = \mathbb{P}_N$$

In practice we can usually ignore N and write $||\phi||$, $||\mathcal{L}|| = \mathbb{P}$, etc.

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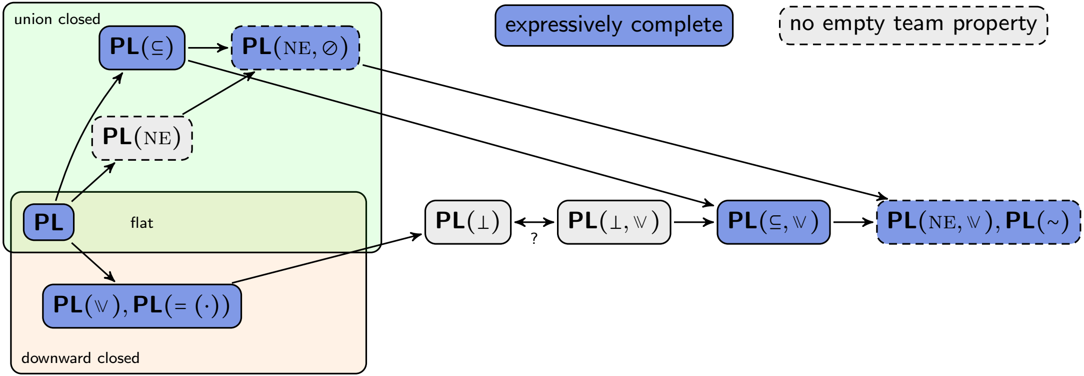
- The proofs of expressive completeness yield normal form for the logics. One can use these to prove the completeness of an axiomatization

Some results in the literature:



$=(\cdot)$: extended dependence atoms: $s \models =\!(\alpha_1, \ldots, \alpha_n, \beta) :\Longleftrightarrow$
$\forall w, w' \in s : (w \models \alpha_i \iff w' \models \alpha_i$ for all $i \in \{1, \ldots, n\})$ implies $w \models \beta \iff w' \models \beta$

$\subseteq$: extended inclusion atoms: $s \models \alpha_1, \ldots \alpha_n \subseteq \beta_1, \ldots, \beta_n :\Longleftrightarrow$
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$\perp$: extended independence atoms: $s \models \alpha_1, \ldots \alpha_n \perp_{\gamma_1, \ldots, \gamma_m} \beta_1, \ldots, \beta_l :\Longleftrightarrow$
$\forall w, w' \in s : (w \models \gamma_i \iff w' \models \gamma_i)$ implies $\exists v \in s : (w \models \alpha_i \iff v \models \alpha_i)$ and $(w' \models \beta_i \iff v \models \beta_i)$ and $(w \models \gamma_i \iff v \models \gamma_i)$

$\oslash$: emptiness operator: $s \models \oslash\phi :\Longleftrightarrow s \models \phi$ or $s = \varnothing$

$\sim$: Boolean negation: $s \models \sim\phi :\Longleftrightarrow s \not\models \phi$

Normal Forms

To show, e.g., that **PL** is expressively complete for the class $\mathbb{F}$ of flat properties—i.e., that $||\mathbf{PL}|| = \mathbb{F}$—one constructs characteristic formulas for flat properties in **PL**.

Characteristic formulas for valuations:

$$\chi_v^N := \bigwedge\{p \mid p \in N, v \models p\} \wedge \bigwedge\{\neg p \mid p \in N, v \not\models p\}$$
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Normal forms

Logic	Normal Form	Type of property characterized
PL	$\bigvee_{s\in\mathcal{P}} \chi_s$	Flat
PL$(\mathbin{\vee\!\!\!\vee})$	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}} \chi_s$	Downward closed, empty team property
PL$(=(\cdot))$	$\bigwedge_{s\in 2^{2^N}\setminus\mathcal{P}_N}(\gamma_s \vee \chi_{2^N\setminus s})$	Downward closed, empty team property
PL$(\subseteq)$	$\bigvee_{s\in\mathcal{P}}(\chi_s \wedge \bigwedge_{v\in s} \top \subseteq \chi_v)$	Union closed, empty team property
PL$(\subseteq, \mathbin{\vee\!\!\!\vee})$	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}}(\chi_s \wedge \bigwedge_{v\in s} \top \subseteq \chi_v)$	Empty team property
PL$(\mathrm{NE}, \oslash)$	$\bigvee_{s\in\mathcal{P}} \oslash \bigvee_{v\in s}(\chi_v \wedge \mathrm{NE})$	Union closed
PL$(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}} \bigvee_{v\in s}(\chi_v \wedge \mathrm{NE})$	All properties

(For the **PL**$(=(\cdot))$-normal form, define $\gamma_0^s := \bot$; $\gamma_1^s := \bigwedge\{=(p) \mid p \in dom(s)\}$ $\gamma_n := \bigvee_1^n \gamma_1$ for $n \geq 2$.)

To show expressive completeness of $\mathbf{PL}(\textsc{ne})$, we consider the following closure property:

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Observe that:

- ϕ is downward closed $\iff$ ϕ is convex and if there is some t s.t. $t \models \phi$, then $\varnothing \models \phi$ (similarly $\mathcal{P}$ is downward closed $\iff$ $\mathcal{P}$ is convex and if $\mathcal{P} \neq \varnothing$, then $\varnothing \in \mathcal{P}$).

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- Let ϕ be upward closed if ($s \models \phi$ and $t \supseteq s$) implies $t \models \phi$.
Then $\phi(N)$ is upward closed $\iff$ ϕ is convex and if there is some t with $dom(t) = N$ s.t. $t \models \phi$, then $2^N \models \phi$.
(similarly $\mathcal{P}$ over N is upward closed $\iff$ $\mathcal{P}$ is convex and if $\mathcal{P} \neq \varnothing$, then $2^N \in \mathcal{P}$)

An example: $q \mathbin{\mathbb{V}} ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$ is not convex:

- $\{w_{pq}\} \models q$ so $\{w_{pq}\} \models q \mathbin{\mathbb{V}} ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$

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Let $\mathbb{CU}$ be the class of convex, union-closed properties.

Theorem (Knudstorp)

$$||\mathbf{PL}(\text{NE})|| = \mathbb{CU}$$

Proof.

$\subseteq$: By induction on ϕ. The only nontrivial case is showing $\psi \vee \chi$ is convex when ψ, χ are union closed convex.

Let $s \vDash \psi \vee \chi$, $t \vDash \psi \vee \chi$, and $s \subseteq u \subseteq t$, where ψ, χ are union closed and convex. Then $s = s_\psi \cup s_\chi$ and $t = t_\psi \cup t_\chi$ where $t_\psi \vDash \psi$, etc. By union closure, $\bigcup||\psi|| \vDash \psi$ and $\bigcup||\chi|| \vDash \chi$. We have $s_\psi \subseteq u \cap \bigcup||\psi|| \subseteq \bigcup||\psi||$ and $s_\chi \subseteq u \cap \bigcup||\chi|| \subseteq \bigcup||\chi||$ so $u \cap \bigcup||\psi|| \vDash \psi$ and $u \cap \bigcup||\chi|| \vDash \chi$ by convexity. $u \subseteq t_\psi \cup t_\chi \subseteq \bigcup||\psi|| \cup \bigcup||\chi||$ so $u \subseteq (u \cap \bigcup||\psi||) \cup (u \cap \bigcup||\chi||)$, whence $u = (u \cap \bigcup||\psi||) \cup (u \cap \bigcup||\chi||)$. Therefore $u \vDash \psi \vee \chi$. □

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$\subseteq$: Let $t_i \in \mathcal{P}$. Then $t_i = \bigcup_{v \in t_i} \{v\}$. For each $v \in t_i$, there is some $\{s_v^j \mid j \in J_v\} \subseteq \overline{\prod \mathcal{P}}$ such that $v \in s_v^j$ and hence also $\{v\} \subseteq s_v^j$ for all $j \in J_v$. Then $\{v\} \vDash \chi_{s_v^j} \wedge \mathrm{NE}$ for all $j \in J_v$ and so $t_i \vDash \bigvee_{v \in t_i} \bigvee_{j \in J_v} (\chi_{s_v^j} \wedge \mathrm{NE})$. For each $s \in \overline{\prod \mathcal{P}}$ there is some $v_i \in s$ such that $v_i \in t_i$ whence $s = s_{v_i}^j$ for some $j \in J_{v_i}$. Therefore $\{s_v^j \mid v \in t_i, j \in J_v\} = \overline{\prod \mathcal{P}}$, and so $t_i \vDash \bigvee_{s \in \overline{\prod \mathcal{P}}} (\chi_s \wedge \mathrm{NE})$.

$\supseteq$: Let $t \vDash \bigvee_{s \in \overline{\prod \mathcal{P}}} (\chi_s \wedge \mathrm{NE})$ so that $t = \bigcup_{s \in \overline{\prod \mathcal{P}}} t_s$ where $t_s \vDash \chi_s \wedge \mathrm{NE}$.
We show $t \subseteq \bigcup \mathcal{P}$; assume for contradiction that $t \nsubseteq \bigcup \mathcal{P}$. Then there is a $v \in t$ such that $v \notin t_i$ for all $t_i \in \mathcal{P}$. Then for any $s \in \overline{\prod \mathcal{P}}$, $v \notin s$. By $t = \bigcup_{s \in \overline{\prod \mathcal{P}}} t_s$ we must have $v \in t_s$ for some $s \in \overline{\prod \mathcal{P}}$, where $t_s \vDash \chi_s \wedge \mathrm{NE}$. But then $v \in t_s \subseteq s$ and $v \notin s$, a contradiction.
We now show $t_i \subseteq t$ for some $t_i \in \mathcal{P}$; assume for contradiction that $t_i \nsubseteq t$ for all $t_i \in \mathcal{P}$. Then for each $t_i \in \mathcal{P}$ there is a $v_i \in t_i$ such that $v_i \notin t$. We have $u := \{v_i \mid t_i \in \mathcal{P}\} \in \overline{\prod \mathcal{P}}$ so $t_s \vDash \chi_u \wedge \mathrm{NE}$ for some $t_s \subseteq t$. Then $t_s \subseteq u$ and $t_s \neq \varnothing$ so there is some $v_i \in t_s \cap u$. But then $v_i \in t_s \subseteq t$ and $v_i \notin t$, a contradiction.
We now have $t_i \subseteq t \subseteq \bigcup \mathcal{P}$. $\bigcup \mathcal{P} \in \mathcal{P}$ by union closure, and therefore $t \in \mathcal{P}$ by convexity. □

Proof.

$\supseteq$: Let $\mathcal{P}_N \in \mathbb{CU}_N$. If $\mathcal{P} = \varnothing$, then it is $\|\bot \wedge \mathrm{NE}\| \in \|\mathbf{PL}(\mathrm{NE})\|$. Otherwise let $\mathcal{P} = \{t_1, \ldots, t_n\}$ ($\mathcal{P} \subseteq 2^{2^N}$ where N is finite, so $\mathcal{P}$ is finite). We show:

$$\mathcal{P} = \|\bigvee\{(\chi_{v_1} \vee \ldots \vee \chi_{v_n}) \wedge \mathrm{NE} \mid (v_1 \times \ldots \times v_n) \in (t_1 \times \ldots \times t_n)\}\| = \|\bigvee_{s \in \overline{\prod \mathcal{P}}} (\chi_s \wedge \mathrm{NE})\|$$

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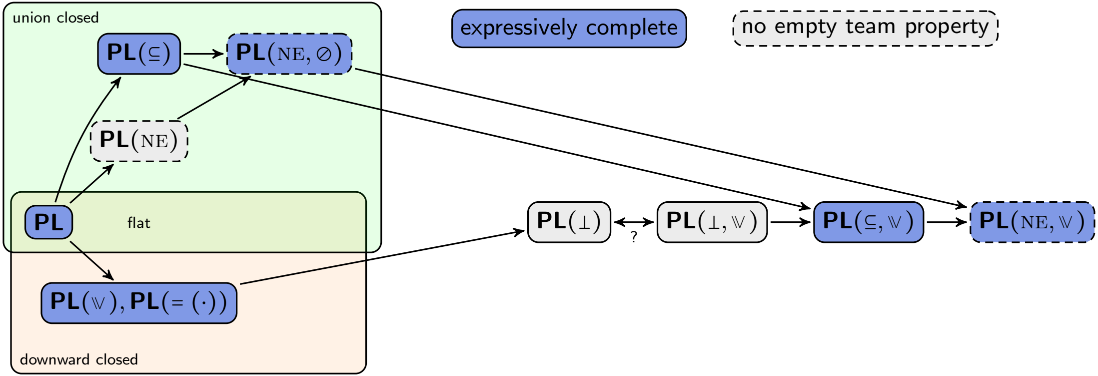
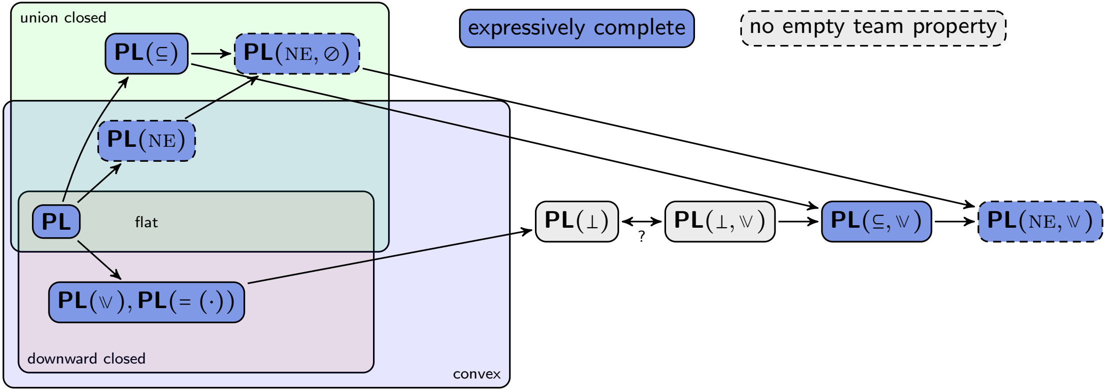
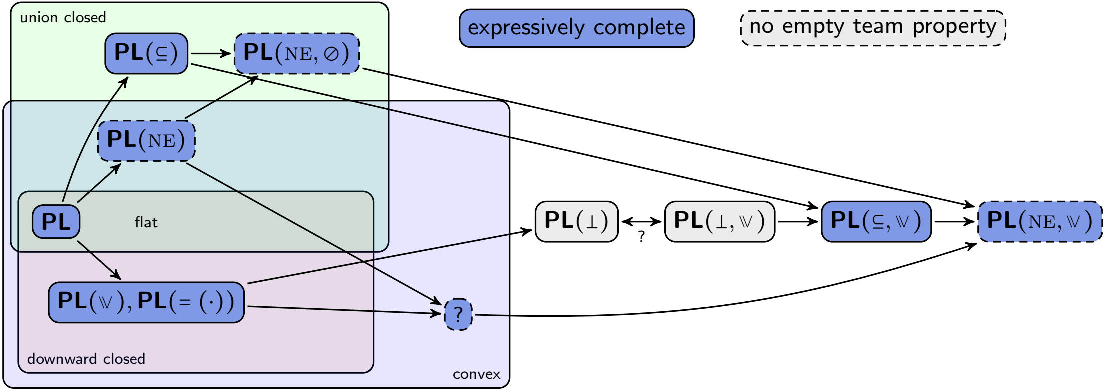
Updated picture:



Updated picture:



Updated picture:



What logic is expressively complete for convex properties? Note that

$$\phi \text{ is convex and has the empty team property } \iff$$
$$\phi \text{ is downward closed and has the empty team property}$$

So $\mathbf{PL}(\mathbb{V})$ and $\mathbf{PL}(=(\cdot))$ are expressively complete for convex properties with the empty team property.

Recall our characteristic formulas for convex union-closed properties:

If $\mathcal{P} \neq \varnothing$: $\mathcal{P} = \| \bigvee_{s \in \overline{\prod \mathcal{P}}} (\chi_s \wedge \mathrm{NE}) \|$

If $\mathcal{P} = \varnothing$: $\mathcal{P} = \| \perp\!\!\!\perp \|$

(where $\perp\!\!\!\perp := \bot \wedge \mathrm{NE}$.) Equivalently we may use:

If $\mathcal{P} \neq \varnothing$: $\mathcal{P} = \| \bigvee_{t \in \mathcal{P}} \chi_t \wedge \bigwedge_{s \in \overline{\prod \mathcal{P}}} ((\chi_s \wedge \mathrm{NE}) \vee \top) \|$

If $\mathcal{P} = \varnothing$: $\mathcal{P} = \| \perp\!\!\!\perp \|$

where $\top := \neg\bot$. Here $\bigvee_{t \in \mathcal{P}} \chi_t$ is a characteristic formula for flat properties, and $\bigwedge_{s \in \overline{\prod \mathcal{P}}} ((\chi_s \wedge \mathrm{NE}) \vee \top)$ is a characteristic formula for upward-closed properties.

To get a characteristic formula for (non-empty) convex properties, simply replace the first conjunct with a characteristic formula for downward-closed properties:

$$\mathcal{P} = \| \mathbb{V}_{t \in \mathcal{P}} \chi_t \wedge \bigwedge_{s \in \overline{\prod \mathcal{P}}} ((\chi_s \wedge \mathrm{NE}) \vee \top) \|$$

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(where $⫫ := \bot \wedge \mathrm{NE}$.) Equivalently we may use:

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If $\mathcal{P} = \varnothing$: $\mathcal{P} = \| ⫫ \|$

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Proposition

For any nonempty convex $\mathcal{P} = \{t_1, \ldots, t_n\}$ over N:

$$\mathcal{P} = \| \bigvee\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t^N \wedge \bigwedge_{s \in \overline{\prod \mathcal{P}}} ((\chi_s^N \wedge \mathrm{NE}) \vee \top) \|_N$$

Proof.

$\subseteq$: Let $t_i \in \mathcal{P}$. Then $t_i \models \chi_{t_i}$ so $t_i \models \bigvee\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t$. If $\mathcal{P}$ has the empty team property, $\overline{\prod \mathcal{P}} = \varnothing$ so the second conjunct is $\top$ (we stipulate $\bigwedge \varnothing := \top$) and we are done. Otherwise let $s \in \overline{\prod \mathcal{P}}$. We have $s = \{v_1, \ldots, v_n\}$ for some $v_1 \in t_1, \ldots, v_n \in t_n$ so there is a $v_i \in s$ such that $v_i \in t_i$. Clearly $t_i \models (\chi_{v_i} \wedge \mathrm{NE}) \vee \top$. Therefore also $t_i \models ((\chi_{v_i} \vee \bigvee_{w \in s \setminus \{v_i\}} \chi_w) \wedge \mathrm{NE}) \vee \top$ whence $t_i \models (\chi_s \wedge \mathrm{NE}) \vee \top$.

$\supseteq$: Let $u \models \bigvee\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t \wedge \bigwedge_{s \in \overline{\prod \mathcal{P}}} ((\chi_s \wedge \mathrm{NE}) \vee \top)$. By $u \models \bigvee\!\!\!\bigvee_{t \in \mathcal{P}} \chi_t$ there is some $t \in \mathcal{P}$ s.t. $u \models \chi_t$ whence $u \subseteq t$.
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This is not surprising given $\mathbf{PL}(\mathrm{NE}, \mathbb{V})$ is complete for all properties, but there is a more general issue with $\vee$: if ϕ or ψ is not union closed, $\phi \vee \psi$ might not be convex.

Let $\mathbb{C}$ be the class of convex properties.

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If $\mathbb{C} \subseteq \|\mathcal{L}\|$ and $\vee$ is uniformly definable in $\mathcal{L}$, then $\|\mathcal{L}\| \notin \mathbb{C}$.

where $\vee$ is uniformly definable in $\mathcal{L}$ if there is a formula $\theta_\vee(p, q) \in \mathcal{L}$ such that $\psi \vee \chi \equiv \theta_\vee(\psi/p, \chi/q)$ for all $\psi, \chi \in \mathcal{L}$. Note that due to failure of uniform substitution in team-based logics, it is possible that $\{\|\psi \vee \chi\| \mid \psi, \chi \in \mathcal{L}\} \subseteq \|\mathcal{L}\|$ without $\vee$ being uniformly definable in $\mathcal{L}$.

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To prove this fact, we recall the intuitionistic implication $\rightarrow$:

$$s \models \phi \rightarrow \psi \iff \forall t \subseteq s : t \models \phi \text{ implies } t \models \psi$$

Consider $\psi := (((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \rightarrow q) \wedge ((r \wedge \mathrm{NE}) \vee \top)$. It is easy to see that $||\psi||$ is convex (the first conjunct is downward closed; the second, upward closed). Note also that $||\psi||$ is not union closed. Let $\theta_\vee$ define $\vee$ in $\mathcal{L}$ and let $\phi := \theta_\vee(\psi, \psi)$. We show $||\phi||$ is not convex. Consider the following team t:

	p	q	r
v_1	1	0	1
v_2	1	0	0
v_3	0	0	1
v_4	0	0	0

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$$s \models \phi \rightarrow \psi \iff \forall t \subseteq s : t \models \phi \text{ implies } t \models \psi$$

Consider $\psi := (((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \rightarrow q) \wedge ((r \wedge \mathrm{NE}) \vee \top)$. It is easy to see that $||\psi||$ is convex (the first conjunct is downward closed; the second, upward closed). Note also that $||\psi||$ is not union closed. Let θ_v define $\vee$ in $\mathcal{L}$ and let $\phi := \theta_v(\psi, \psi)$. We show $||\phi||$ is not convex. Consider the following team t:

	p	q	r
v_1	1	0	1
v_2	1	0	0
v_3	0	0	1
v_4	0	0	0

We have $\forall t \subseteq \{v_1, v_2\} : t \not\models (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})$ so $\{v_1, v_2\} \models ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \rightarrow q$. Also $\{v_1, v_2\} \models (r \wedge \mathrm{NE}) \vee \top$ so $\{v_1, v_2\} \models \psi$. Similarly $\{v_3, v_4\} \models \psi$. Therefore $t \models \psi \vee \psi$ so $t \models \phi$. Clearly also $\{v_3\} \models \psi$ so $\{v_3\} \models \phi$. Now assume for contradiction that $\{v_2, v_3\} \models \phi$. Then $\{v_2, v_3\} = t_1 \cup t_2$ where $t_1 \models \psi$ and $t_2 \models \psi$. We cannot have $t_i = \{v_2\}$ because clearly $\{v_2\} \not\models \psi$. So one of the subteams t_i must be $\{v_2, v_3\}$. But $\{v_2, v_3\} \not\models ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \rightarrow q$. Since $\{v_3\} \subseteq \{v_2, v_3\} \subseteq t$, ϕ is not convex.

To obtain a logic expressively complete for $\mathcal{L}$, we must therefore change the classical base of the logic.

Syntax of classical propositional logic with $\rightarrow$ **PL**$_{\rightarrow}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \rightarrow \alpha$$

where $p \in Prop$ and $s \models \bot \iff s = \varnothing$. As with **PL**, it can be shown **PL**$_{\rightarrow}$ is flat and its semantics correspond to standard semantics on singletons.

Syntax of **PL**$_{\rightarrow}(\nabla)$:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \rightarrow \phi \mid \nabla\phi$$

∇ is an "epistemic might" operator which has been used to formalize epistemic contradictions:

$$s \models \nabla\phi \iff \exists t \subseteq s : t \neq \varnothing \text{ and } t \models \phi$$

Epistemic contradiction: #It is raining but it might not be raining.

Formalized as: $r \wedge \nabla\neg r$. Contradiction: $r \wedge \nabla\neg r \models \perp\!\!\!\perp$.

Note that $\nabla\phi \equiv (\phi \wedge \text{NE}) \vee \top$ and that $\text{NE} \equiv \nabla\top$.

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Proposition

$\|\mathbf{PL}_{\rightarrow}(\nabla)\| \subseteq \mathbb{C}$ (i.e., $\mathbf{PL}_{\rightarrow}(\nabla)$ is convex).

Proof.

$p, \bot$ are flat and and hence convex. $\phi \rightarrow \psi$ is downward closed and hence convex. $\nabla\phi$ is upward closed and hence convex. The conjunction case follows immediately from the induction hypothesis. □

To show $\mathbb{C} \subseteq \|\mathbf{PL}_{\rightarrow}(\nabla)\|$, it suffices by what we have shown above to show that

- $\| \perp\!\!\!\perp \| \in \|\mathbf{PL}_{\rightarrow}(\nabla)\|$
- for all non-empty $\mathcal{P} \in \mathbb{C}$, $\| \bigvee\!\!\!\bigvee_{t \in \mathbf{P}} \chi_t \wedge \bigwedge_{s \in \overline{\prod \mathbf{P}}} ((\chi_s \wedge \mathrm{NE}) \vee \top)\| \in \|\mathbf{PL}_{\rightarrow}(\nabla)\|$

We have $\| \perp\!\!\!\perp \| = \|\nabla\bot\| \in \|\mathbf{PL}_{\rightarrow}(\nabla)\|$. We also have

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$$\bigvee\!\!\!\bigvee_{i\in I} \alpha_i := \bigwedge_{i\in I}((\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \to \alpha_i)$$

E.g., $\alpha \bigvee\!\!\!\bigvee \beta = (\nabla\neg\alpha \to \beta) \wedge (\nabla\neg\beta \to \alpha)$

Lemma

$t \models \bigvee\!\!\!\bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \models \alpha_i$.

Proof.

$\implies$: Assume for contradiction that for each $i \in I$, $t \not\models \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \models \neg\alpha_i$. Then for each $i \in I$, $t \models \nabla\neg\alpha_i$. By $t \models (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

$\impliedby$: Let $t \models \alpha_i$. Let $s \subseteq t$ be such that $s \models \bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j$. By downward closure also $s \models \alpha_i$. So $t \models (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \to \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \models \bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j$ because $s \models \alpha_i$. Therefore $t \models (\bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j) \to \alpha_k$. □

Theorem

$$\|\mathbf{PL}_{\to}(\nabla)\| = \mathbb{C}$$

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$t \vDash \bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \vDash \alpha_i$.

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$\Longrightarrow$: Assume for contradiction that for each $i \in I$, $t \not\models \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \models \neg \alpha_i$. Then for each $i \in I$, $t \models \nabla \neg \alpha_i$. By $t \models (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_i) \to \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

$\Longleftarrow$: Let $t \models \alpha_i$. Let $s \subseteq t$ be such that $s \models \bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_j$. By downward closure also $s \models \alpha_i$. So $t \models (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_j) \to \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \models \bigwedge_{j \in I \setminus \{k\}} \nabla \neg \alpha_j$ because $s \models \alpha_i$. Therefore $t \models (\bigwedge_{j \in I \setminus \{k\}} \nabla \neg \alpha_j) \to \alpha_k$. □

Theorem

$$||\mathbf{PL}_{\to}(\nabla)|| = \mathbb{C}$$

We define:

$$\bigvee_{i\in I} \alpha_i := \bigwedge_{i\in I}\Big(\big(\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j\big) \to \alpha_i\Big)$$

E.g., $\alpha \vee \beta = (\nabla\neg\alpha \to \beta) \wedge (\nabla\neg\beta \to \alpha)$

Lemma

$t \models \bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \models \alpha_i.$

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$\implies$: Assume for contradiction that for each $i \in I$, $t \not\models \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \models \neg\alpha_i$. Then for each $i \in I$, $t \models \nabla\neg\alpha_i$. By $t \models (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

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Theorem

$$||\mathbf{PL}_{\to}(\nabla)|| = \mathbb{C}$$

We define:

$$\bigvee_{i\in I} \alpha_i := \bigwedge_{i\in I}((\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \to \alpha_i)$$

E.g., $\alpha \vee \beta = (\nabla\neg\alpha \to \beta) \wedge (\nabla\neg\beta \to \alpha)$

Lemma

$t \vDash \bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \vDash \alpha_i$.

Proof.

$\implies$: Assume for contradiction that for each $i \in I$, $t \nvDash \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$. Then for each $i \in I$, $t \vDash \nabla\neg\alpha_i$. By $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \vDash \alpha_i$.

$\impliedby$: Let $t \vDash \alpha_i$. Let $s \subseteq t$ be such that $s \vDash \bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j$. By downward closure also $s \vDash \alpha_i$. So $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \to \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \vDash \bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j$ because $s \vDash \alpha_i$. Therefore $t \vDash (\bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j) \to \alpha_k$. □

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$$\|\mathbf{PL}_{\to}(\nabla)\| = \mathbb{C}$$

We define:

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Lemma

$t \models \bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \models \alpha_i.$

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$\Longrightarrow$: Assume for contradiction that for each $i \in I$, $t \not\models \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \models \neg\alpha_i$. Then for each $i \in I$, $t \models \nabla\neg\alpha_i$. By $t \models (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

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Theorem

$$\|\mathbf{PL}_{\rightarrow}(\nabla)\| = \mathbb{C}$$

We define:

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E.g., $\alpha \bigvee\!\!\!\!\bigvee \beta = (\nabla \neg \alpha \to \beta) \land (\nabla \neg \beta \to \alpha)$

Lemma

$t \models \bigvee\!\!\!\!\bigvee_{i \in I} \alpha_i \iff \exists i \in I : t \models \alpha_i$.

Proof.

$\Longrightarrow$: Assume for contradiction that for each $i \in I$, $t \not\models \alpha_i$. By flatness, for each $i \in I$ there is some $v_i \in t$ with $v_i \models \neg \alpha_i$. Then for each $i \in I$, $t \models \nabla \neg \alpha_i$. By $t \models (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_i) \to \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

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Theorem

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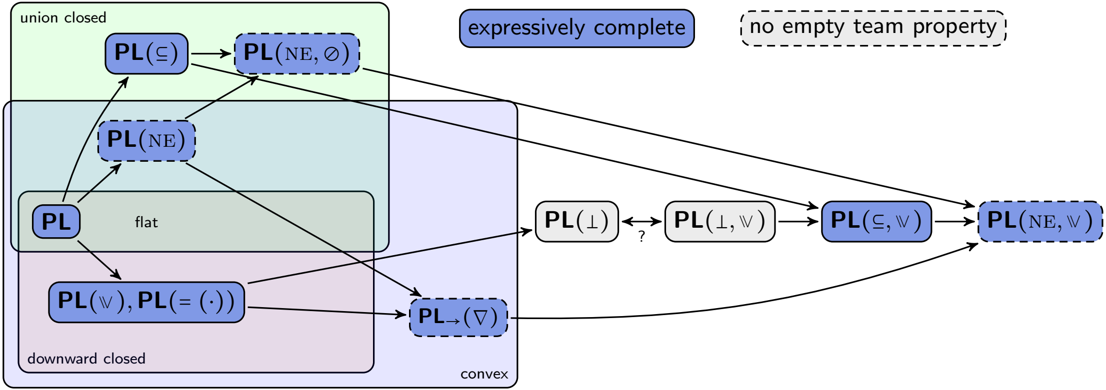
Updated picture:



Normal forms

Logic	Normal Form	Type of property characterized
PL	$\bigvee_{s\in\mathcal{P}} \chi_s$	Flat
PL($\mathbin{\vee\!\!\!\vee}$)	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}} \chi_s$	Downward closed, empty team property
PL$(=(\cdot))$	$\bigwedge_{s\in 2^{2^N}\setminus\mathcal{P}_N}(\gamma_s \vee \chi_{2^N\setminus s})$	Downward closed, empty team property
PL($\subseteq$)	$\bigvee_{s\in\mathcal{P}}(\chi_s \wedge \bigwedge_{v\in s} \top \subseteq \chi_v)$	Union closed, empty team property
PL($\subseteq$, $\mathbin{\vee\!\!\!\vee}$)	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}}(\chi_s \wedge \bigwedge_{v\in s} \top \subseteq \chi_v)$	Empty team property
PL$(\mathrm{NE}, \oslash)$	$\bigvee_{s\in\mathcal{P}} \oslash \bigvee_{v\in s}(\chi_v \wedge \mathrm{NE})$	Union closed
PL$(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$	$\mathbin{\vee\!\!\!\vee}_{s\in\mathcal{P}} \bigvee_{v\in s}(\chi_v \wedge \mathrm{NE})$	All properties
PL(NE)	$\bigvee_{s\in\overline{\prod\mathcal{P}}}(\chi_s \wedge \mathrm{NE})$	Convex, union closed
PL(NE)	$\bigvee_{t\in\mathcal{P}} \chi_t \wedge \bigwedge_{s\in\overline{\prod\mathcal{P}}}((\chi_s \wedge \mathrm{NE}) \vee \top)$	Convex, union closed
	$\bigwedge_{s\in\overline{\prod\mathcal{P}}}((\chi_s \wedge \mathrm{NE}) \vee \top)$	Upward closed
$\mathbf{PL}_{\rightarrow}(\nabla)$	$\mathbin{\vee\!\!\!\vee}_{t\in\mathcal{P}} \chi_t \wedge \bigwedge_{s\in\overline{\prod\mathcal{P}}}((\chi_s \wedge \mathrm{NE}) \vee \top)$	Convex

(For the **PL**$(=(\cdot))$-normal form, define $\gamma_0^s := \bot$; $\gamma_1^s := \bigwedge\{=(p) \mid p \in dom(s)\}$ $\gamma_n := \bigvee_1^n \gamma_1$ for $n \geq 2$.)
(In $\mathbf{PL}_{\rightarrow}(\nabla)$, $\mathbin{\vee\!\!\!\vee}_{i\in I} \alpha_i := \bigwedge_{i\in I}((\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \rightarrow \alpha_i)$.)

Relationship with inquisitive logic: *InqB*, propositional inquisitive logic, has the syntax:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \mathbin{\mathbb{V}} \phi$$

InqB is expressively complete for downward-closed properties with the empty state property, so $||InqB|| \subset ||\mathbf{PL}_{\to}(\nabla)||$. $\mathbb{V}$ is not uniformly definable in general in $\mathbf{PL}_{\to}(\nabla)$ since $\mathbf{PL}_{\to}(\nabla, \mathbb{V})$ is not convex.

Similar logics which are either not convex or cannot express all convex properties:

- $PL_{\to}(\mathbb{V}, \nabla)$ (propositional inquisitive logic with ∇) is not convex. Example: $(p \wedge \nabla q) \mathbin{\mathbb{V}} (a \wedge \nabla b)$.
- $PL_{\to}(\mathrm{NE})$ is not complete for convex properties because it is "downward closed modulo the empty team": $s \models \phi$ and $t \subseteq s$ where $t \neq \varnothing$ imply $t \models \phi$. Similarly for $PL_{\to}(\mathrm{NE}, \mathbb{V})$.

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